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THEORY OF A FEEDBACK TO CURE TRANSVERSE MODE COUPLING INSTABILITY

Budker INP 2001–37

Novosibirsk
2001

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Abstract

In the present paper an attempt is made to develop an advanced transverse feedback theory capable to clarify the conditions, at which the resistive feedback can cure the TMC instability. The hollow beam model is used for analysis because of its simplicity. As it appeared an important role can play chromaticity. Negative chromaticity combined with resistive feedback as from theory follows can increase the TMC instability threshold by several times. In any case, the basic results obtained with the hollow beam model may be then corrected using more complicated one.

1. Introduction

Transverse Mode Coupling Instability (TMCI) is the main limiting factor for bunch intensity in large storage rings. Its basic features are well familiar due to many theoretical works (see, for example, [1–5]).

One has proposed to cure this instability with the reactive feedback that would prevent the zero mode frequency from changing with increasing the beam intensity [6]. In [7,8] a theory of reactive feedback has been developed in two particle approach and with Vlasov equation. Theory shows that the reactive feedback should appreciably increase the threshold of TMCI. Such conclusion has been confirmed by simulation [9,10]. On the contrary, the resistive feedback was found to be «completely ineffective as a cure for the transverse mode coupling instability» [8].

An action of a feedback on the threshold of TMCI was later on examined experimentally at PEP [11]. It was confirmed that a reactive feedback really is capable to increase the TMCI threshold noticeably.

But it turned out unexpectedly that resistive one can increase this threshold also and even more effectively [11]. An analogous result was obtained in experiments at VEPP-4 (BINP, Novosibirsk) [12].

For a long time the efforts were ineffective at LEP (CERN) to increase the beam current, TMCI limited, by application of a feedback. V.Danilov and E.Perevedentsev [13] explaining such failure have postulated that transverse feedback could not work correctly when the bunch is passing some amount of the transverse impedance between a pick up and a kicker as it was at LEP. To overcome this circumstance they have proposed to introduce in the feedback chain an artificial oscillator that should model the beam behavior between the pick up and a kicker and thus give a correct kick. After extensive simulation [14] and some improvements the feedback was tested at LEP [15]. Certain increase of the threshold beam current (near 5% instead of 10 % calculated) was accompanied by more stable behavior of the beam [15].

From what was said it is evident that not all is clear up to now with a feedback acting against TMC instability. However the knowledge of its properties is important because now are discovered new mechanisms of its origin (not only wake fields), in particular, interaction with electron clouds [16] and coherent beam – beam interaction.

In the present paper an attempt is made to develop an advanced transverse feedback theory capable to clarify the conditions, at which the resistive feedback can cure TMC instability. The hollow beam model was used for analysis because of its simplicity. In this approach «radial modes» could not be taken into account. In any case, the basic results obtained with the hollow beam model may be then corrected using more complicated one.

In Section 2 the mode equations are derived with the account of chromaticity. These equations allow one to analyze the simplest reactive feedback.

But for the more detailed analysis equations describing the feedback are derived in Section 3. It is shown that the feedback signal should be so delayed that the bunch and the signal arrive to the kicker simultaneously.

In Section 4 an analysis of stability with the feedback is accomplished. As it turned out the combined application of resistive feedback and negative chromaticity permits one to provide stability in wide limits of the feedback phases.

2. Mode equations

Our goal in this section is to obtain differential mode equations that describe time evolution of coupled modes. With the help of these equations the problem of stability can be reduced to the eigenvalue problem.

We start from equations of transverse motion of single particle, more precisely, of macroparticle, under the action of the Lorentz force. Then we consider several macroparticles in one separatrix, uniformly distributed over synchrotron oscillation phases. The motion of these macroparticles can be represented in the form of symmetrical modes.

The equations for particle transverse oscillations in action–angle variables were derived in [17]:

$$\dot{J} = e\overline{\frac{\partial x}{\partial \psi}(E_x - vB_y)}, \ \ \dot{\varphi} = \dot{\psi} - \Omega = -e\overline{\frac{\partial x}{\partial J}(E_x - vB_y)}. \tag{1}$$

Here J, ψ are action–angle variables of particle transverse oscillations, φ is slowly varying phase, x is transverse particle coordinate, Ω is an angular frequency of oscillations. The Lorentz force $e(E_x - v \cdot B_y)$ should be written in the accompanying system of coordinates, passage to which is performed by formal transformation $l = z + \omega_0 \cdot R \cdot t$, where l is longitudinal coordinate in laboratory system, z is longitudinal coordinate in accompanying system, ω_0 is an angular revolution frequency of equilibrium particle, R is an average radius of the ring. The bars over right hand side of equations mean averaging over fast time (over several periods of betatron oscillations).

If a particle performs longitudinal (synchrotron) oscillations then betatron frequency can change periodically due to chromaticity. In this case $\Omega = \Omega(p_z)$, p_z being momentum deviation from equilibrium momentum p_0: $p = p_0 + p_z$. The frequency of transverse oscillations depending on p_z is in this case

$$\Omega(p_z) \cong \Omega_o + \frac{\partial\Omega}{\partial p_z} \cdot p_z, \tag{2}$$

where Ω_0 is betatron frequency of synchronous particle ($p_z = 0$). As $\Omega = \nu \cdot \omega_0$, ω_0 being instantaneous revolution frequency, then

$$\frac{\partial\Omega}{\partial p_z} = \nu \cdot \frac{\partial\omega}{\partial p_z} + \omega \cdot \frac{\partial\nu}{\partial p_z}. \tag{3}$$

Instantaneous revolution frequency is

$$\omega = \omega_0 \left(1 - \alpha \cdot \frac{p_z}{p_0}\right), \tag{4}$$

therefore

$$p_z \cdot \frac{\partial\omega}{\partial p_z} = -\omega_0 \cdot \alpha \cdot \frac{p_z}{p_0}. \tag{5}$$

Here α is a momentum compaction factor.

Substituting (4) and (5) into (3) and then into (2) we obtain

$$\Omega(p_z) \cong \Omega_0 + \frac{\omega_0\alpha}{p_0} \cdot \left(\frac{p_0}{\alpha} \cdot \frac{\partial\nu}{\partial p_z} - \nu\right) \cdot p_z.$$

Let us denote $\kappa = \frac{p_0}{\alpha} \cdot \frac{\partial\nu}{\partial p}$. This parameter is $1/\alpha$ time more than commonly used ν prime: $\nu' = p_0 \cdot \frac{\partial\nu}{\partial p_z}$. Taking into account that $p_z = -\frac{m_s}{\alpha} \cdot \dot{z}$, ($m_s$ is mass of synchronous particle) the expression obtained above can be reduced to

$$\Omega(p_z) \cong \Omega_0 + \frac{\nu - \kappa}{R} \cdot \dot{z}, \tag{6}$$

here z is a longitudinal coordinate read from synchronous particle.

Integrating (6) over time at constant frequency we get

$$\psi = \Omega_0 \cdot t + \frac{\nu - \kappa}{R} \cdot z + \varphi, \tag{7}$$

where φ is slowly varying phase; or denoting $\xi = \nu - \kappa$

$$\psi = \Omega_0 \cdot t + \xi \cdot \frac{z}{R} + \varphi. \tag{8}$$

Further we use complex amplitude of oscillations

$$y = a \cdot e^{j\varphi}, \tag{9}$$

where a is modulus (amplitude), φ is a phase.

Note that an amplitude for linear or close to linear oscillations can be expressed through an action

$$a = \sqrt{\frac{2J}{m_s \cdot \Omega}},$$

where m_s is a mass of synchronous particle.

The derivative of the complex amplitude with respect to (slow) time is

$$\dot{y} = (\dot{a} + j \cdot a \cdot \dot{\varphi}) \cdot e^{j\varphi}.$$

We find from expression for the amplitude

$$\dot{a} = \sqrt{\frac{2}{m_s \cdot \Omega}} \cdot \frac{\dot{J}}{2\sqrt{J}}.$$

Substituting into $\dot{y}$ one obtains

$$\dot{y} = \left(\sqrt{\frac{2}{m_s}} \cdot \frac{\dot{J}}{2\sqrt{J}} + j \cdot \dot{\varphi} \cdot \sqrt{\frac{2J}{m_s.\Omega}} \right) \cdot e^{j\varphi}. \tag{10}$$

On the other hand, substituting into (10) expression (1) for $\dot{J}$ and $\dot{\varphi}$ we obtain

$$\dot{y} = \left(\sqrt{\frac{2}{m_s \cdot \Omega}} \cdot \frac{1}{2\sqrt{J}} \cdot e \, \overline{\frac{\partial x}{\partial \psi} (E_x - vB_y)} - j \cdot \sqrt{\frac{2J}{m_s \cdot \Omega}} \cdot e \, \overline{\frac{\partial x}{\partial J} \cdot (E_x - vB_y)} \right) \cdot e^{j\varphi}.$$

Derivatives necessary for evaluation of averages are

$$x = \sqrt{\frac{2J}{m_s \cdot \Omega}} \cdot \sin\psi, \qquad \frac{\partial x}{\partial \psi} = \sqrt{\frac{2J}{m_s \cdot \Omega}} \cdot \cos\psi, \qquad \frac{\partial x}{\partial J} = \sqrt{\frac{2}{m_s \cdot \Omega}} \cdot \frac{1}{2\sqrt{J}} \cdot \sin\psi.$$

Substituting we get

$$\dot{y} = \left(\frac{e}{m_s \cdot \Omega} \cdot \overline{\cos\psi \cdot (E_x - vB_y)} - j \frac{e}{m_s \cdot \Omega} \cdot \overline{\sin\psi \cdot (E_x - vB_y)} \right) \cdot e^{j\varphi} =$$

$$= \frac{e}{m_s \cdot \Omega} \cdot \overline{e^{-j\psi} \cdot (E_x - vB_y)} \cdot e^{j\varphi}.$$

Since φ is slowly varying variable, the exponent $e^{j\varphi}$ can be inserted under the averaging bar and then we get

$$\dot{y} = \frac{e}{m_s \cdot \Omega} \cdot \overline{e^{-j(\psi-\varphi)} \cdot (E_x - vB_y)}\,.$$

As was stated above

$$\psi - \varphi = \Omega_0 \cdot t + \xi \cdot \frac{z}{R}\,,$$

so substituting one gets

$$\dot{y} = \frac{e}{m_s \cdot \Omega} \cdot \overline{e^{-j(\Omega_0 t + \xi \cdot z / R)} (E_x - vB_y)}\quad . \tag{11}$$

Up to this point we considered one macroparticle and field acting on it. But in more general case the number of macroparticles can be more than one, say h. Each particle is acted upon by electromagnetic field at the point where it is situated. This electromagnetic field is a sum of fields induced by each macroparticle. According to what has been said, the last expression can be written in the following form:

$$\dot{y}_f = \frac{e}{m_s \cdot \Omega} \cdot \overline{e^{-j(\Omega_0 t + \xi \cdot z_f / R)} (E_x - vB_y)_f}\,, \tag{12}$$

where the subscript f is the ordinal number of the given macroparticle, $f = 1,2,...,h$. The field $(E_x – v{\cdot}B_y)_f$ is a sum of the fields created by all macroparticles at the point where the fth one is situated.

Each macroparticle is characterized besides of a complex amplitude of transverse oscillations also by amplitude and phase of synchrotron oscillations. In what follows amplitudes of synchrotron oscillations of all macroparticles are assumed to be equal. Presupposed is also that macroparticles are uniformly distributed over the phases of synchrotron oscillations, i.e. a synchrotron phase of the fth particle is $2\pi f / h$.

We will consider cavities with zero transverse electric field at the axis (at the beam region). This assumption is valid for real cavities and allows us to simplify Lorentz force to magnetic one only.

According to [17] magnetic field in a cavity with zero transverse electric field can be written in the form

$$vB_y = -\sum_{k,m} e^{jmz/R} \cdot L^{-1}\left\{\omega_0 R \cdot \frac{\partial E_{kz,m}}{\partial x} \cdot \frac{Z_k(s - jm\omega_0)}{s - jm\omega_0} \cdot I_{km}(s)\right\} \tag{13}$$

in the synchronous particle rest system.

Here L^{-1} is inverse Laplase transformation with complex variable s, $E_{kz,m}$ is the mth Fourier harmonic (along the orbit) of the longitudinal electric field of the kth cavity mode at the beam axis, $Z_k(s)$ is an impedance of the kth cavity mode:

$$Z_k(s) = \frac{s}{(s^2 - s_k^2)\cdot\varepsilon\cdot\int |E_k|^2 dV},$$

$I_{km}(s)$ is mth induced current harmonic of kth cavity mode. Summation over k is made over all cavity modes with nonzero $\partial E_{kz,m}/\partial x$ and over m within the limits $\pm\infty$.

The current corresponding to one macroparticle (of the number f') can be represented in a form [17]

$$I_{kmf'}(s) = e\cdot\frac{N}{h}\cdot L\left\{e^{-jmz_{f'}/R}\cdot x_{f'}\cdot\frac{\partial E_{kz,-m}}{\partial x}\right\} \tag{14}$$

were $x_{f'} = \sqrt{\frac{2J_{f'}}{m_s\cdot\Omega}}\cdot\sin\psi$ is transverse coordinate of macroparticle, which is a source of electromagnetic field, N is number of particles in a bunch, N/h is number of particles in a macroparticle. Note that $z_{f'}(t)$ and $x_{f'}(t)$ are time functions. For f' th macroparticle

$$x_{f'} = \sqrt{\frac{2J_{f'}}{m_s\cdot\Omega}}\cdot\sin\left(\Omega_0 + \xi\cdot\frac{z_{f'}}{R} + \varphi_{f'}\right) . \tag{15}$$

Total magnetic field is determined by summarized current

$$I_{km} = e\cdot\frac{N}{h}v\cdot\sum_{f'}\frac{\partial E_{kz,-m}}{\partial x}\cdot L\{e^{-jmz_{f'}/R}\cdot x_{f'}\}.$$

Substituting this current into (13) we obtain

$$vB_y = -e\cdot\frac{N}{h}v\cdot\omega_0 R\cdot\sum_{k,m}\frac{\partial E_{kz,m}}{\partial x}\cdot\frac{\partial E_{kz,-m}}{\partial x}\cdot\sum_{f'}e^{jmz_f/R}\cdot e^{-jmz_{f'}/R}\times$$
$$\times L^{-1}\left\{\frac{Z_k(s - jm\omega_0)}{s - jm\omega_0}\cdot L\{x_{f'}(t)\}\right\} .$$

Here exponent $e^{-jmz_{f'}/R}$ is taken out from the symbol of inverse Laplace transformation because it is varying with (slow) synchrotron frequency and can be considered as constant with respect to fast betatron time.

In what follows we'll use symbol Ω instead of Ω_0.

Using last expression in (12) yields

$$\dot{y}_f = e^2 \frac{N}{h} v \cdot \frac{\omega_0 R}{m_s \Omega} \cdot \sum_{k,m} \frac{\partial E_{kz,m}}{\partial x} \cdot \frac{\partial E_{kz,-m}}{\partial x} \cdot \sum_{f'} e^{j(m-\xi) z_f / R} \cdot e^{-jm z_{f'} / R} \times$$
$$\times \overline{e^{-j\Omega t} \cdot L^{-1} \left\{ \frac{Z_k (s - jm\omega_o)}{s - jm\omega_0} \cdot L\{x_{f'}(t)\} \right\}}. \tag{16}$$

Laplace transform of $x_{f'}(t)$ can be calculated as follows

$$L\{x_{f'}\} = L\{a_{f'} \cdot \sin(\Omega t + \xi \cdot \frac{z}{R} + \varphi_{f'})\} = \frac{a_{f'}}{2j} \cdot \left(\frac{e^{j(\xi \cdot z / R + \varphi_{f'})}}{s - j\Omega} - \frac{e^{-j(\xi \cdot z / R + \varphi_{f'})}}{s + j\Omega} \right).$$

Here amplitude $a_{f'}$ is considered as slow variable and therefore is taken out from the symbol of Laplace transformation.

The average in (16) is

$$\overline{e^{-j\Omega t} \cdot L^{-1} \left\{ \frac{Z_k (s - jm\omega_o)}{s - jm\omega_0} \cdot L\{x_{f'}(t)\} \right\}} =$$
$$= a_{f'} \cdot \overline{L^{-1} \cdot \left\{ \frac{Z_k (s - jm\omega_0 + j\Omega)}{s - jm\omega_0 + j\Omega} \cdot \frac{1}{2j} \cdot \left(\frac{e^{j(\xi z / R + \varphi_{f'})}}{s} - \frac{e^{-j(\xi \cdot z_{f'} / R + \varphi_{f'})}}{s + 2j\Omega} \right) \right\}}.$$

For calculation of the average we'll use following relation

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) \cdot dt = \lim_{s \to +0} s \cdot F(s),$$

where *F(s) = L{f(t)}* is Laplace transform of *f(t).*

As a result we obtain

$$\overline{e^{-j\Omega t} \cdot L^{-1} \left\{ \frac{Z_k (s - jm\omega_0)}{s - jm\omega_0} \cdot L\{x_{f'}(t)\} \right\}} = \frac{1}{2j} \cdot a_{f'} \cdot e^{j(\xi \cdot z / R + \varphi_{f'}\}} \times$$
$$\times \frac{Z_k (-jm\omega_0 + j\Omega)}{-jm\omega_0 + j\Omega} = \frac{1}{2j} \cdot y_{f'} \cdot e^{j\xi \cdot z / R} \cdot \frac{Z_k (-jm\omega_0 + j\Omega)}{-jm\omega_0 + j\Omega}$$

and substituting this into equation (16) yields

$$\dot{y}_f = -\frac{1}{2} e^2 \frac{N}{h} v \frac{\omega_0 R}{m_s \Omega} \cdot \sum_{f'} y_{f'} \sum_{k,m} \frac{\partial E_{kz,m}}{\partial x} \cdot \frac{\partial E_{kz,-m}}{\partial x} \times$$

$$\times \frac{Z_k(-jm\omega_0 + j\Omega)}{-jm\omega_0 + j\Omega} \cdot e^{j(m-\xi)\cdot z_f / R} \cdot e^{-j(m-\xi)\cdot z_{f'} / R} \quad .$$

As z_f and $z_{f'}$ explicitly depend on time obtained differential equation set for complex amplitudes y_f is a set with time‑varying coefficients.

It is profitable to go to new variables by discrete Fourier transformation

$$y_f = \sum_n u_n \cdot e^{j\frac{2\pi f}{h}n} \, , \qquad (18)$$

u_n being new variables, h is a number of macroparticles in a separatrix. There exists an inverse transformation also

$$u_n = \frac{1}{h} \cdot \sum_f y_f \cdot e^{-j\frac{2\pi f}{h}f} \, . \qquad (19)$$

Summation in sums (18) and (19) should be made over all Fourier harmonics or over all macroparticles.

The sense of this transformation is a transition to description of macroparticles motion by symmetrical modes instead of complex amplitudes of separate macroparticles. Such transition is advantageous if one keeps in mind that important role will play only a few lower modes. Furthermore, as we'll see later, transition to symmetrical modes allows one to get rid of time‑varying coefficients in the equation set.

It is convenient to enumerate the modes symmetrically with respect to zero:

$$n = -\frac{h-1}{2}, \ldots, \quad 0, \ldots \frac{h-1}{2} \, .$$

Then zero mode n = 0 corresponds to oscillation of the center of gravity of a bunch.

Differentiating (19) with respect to slow time we obtain

$$\dot{u}_n = \sum_f \dot{y}_f \cdot e^{-j\frac{2\pi n}{h}f} \, . \qquad (20)$$

Now let us multiply left and right side of each equation (18) by

$$\frac{1}{h} \cdot \exp(-j\frac{2\pi}{h}f)$$

and then add all equations. Change simultaneously $y_{f'}$ in right hand side by $u_{n'}$ with Fourier transform. As a result we obtain

$$\dot{u}_n = -\frac{1}{2} e^2 \frac{N}{h} v \cdot \frac{\omega_0 R}{m_s \Omega} \cdot \sum_{n'} u_{n'} \sum_{k,m} \frac{\partial E_{kz,m}}{\partial x} \frac{\partial E_{kz,-m}}{\partial x} \frac{Z_k(-jm\omega_0 + j\Omega)}{-m\omega_0 + jm\Omega} \times$$

(21)

$$\times \sum_f \exp\left[\frac{j(m-\xi)z_f}{R}\right] \cdot \exp\left(-j\frac{2\pi n}{h} f\right) \cdot \sum_{f'} \exp\left[-\frac{j(m-\xi)z_{f'}}{R}\right] \cdot \exp\left(j\frac{2\pi n'}{h} f'\right)$$

the set of differential equations for variables u_n.

Next let us transform the last two sums in (21). As

$$z = \sigma_z(J_z) \cdot \sin\psi_z,$$

then

$$\exp[j(m-\xi)z / R = \sum_s A_{m-\xi,s}(J_z) \cdot \exp(js\psi_z), \tag{22}$$

where

$$A_{m-\xi,s} \cdot (J_z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left[j\left(\frac{m-\xi}{R} \cdot \sigma_z \sin\psi_z - s \cdot \psi_z\right)\right] \cdot d\psi_z . \tag{23}$$

Here J_z, ψ_z are variables action-angle for a longitudinal motion.

The last integral is nothing but Bessel function

$$A_{m-\xi,s} \cdot (J_z) = J_s\left(\frac{m-\xi}{R} \cdot \sigma_z\right) = J_s\left(\frac{m-\xi}{R} \cdot \sqrt{\frac{2J_z}{M \cdot \Omega}}\right).$$

As longitudinal oscillations are distributed uniformly over phase then

$$\exp[j(m-\xi)z_f / R] = \sum_s A_{m-\xi,s} \cdot (J_z) \cdot \exp\left[js\left(\psi_z + \frac{2\pi f}{h}\right)\right]$$

and, analogously

$$\exp[-j(m-\xi)z_{f'} / R] = \sum_{s'} A_{-(m-\xi),s'} \cdot (J_z) \cdot \exp\left[js'\left(\psi_z + \frac{2\pi f'}{h}\right)\right].$$

Here

$$A_{-(m-\xi),s'}\cdot(J_z)=J_{s'}\left(-\frac{m-\xi}{R}\sigma_z\right)=J_{s'}\left(-\frac{m-\xi}{R}\sqrt{\frac{2J_z}{M\cdot\Omega_z}}\right).$$

Substituting these expressions into (21) we obtain

$$\sum_f \exp\left[\frac{j(m-\xi)z_f}{R}\right]\cdot\exp\left[-\frac{j2\pi n}{h}f\right]=$$

$$=\sum_s A_{m-\xi,s}\cdot(J_z)\cdot e^{js\Psi_z}\cdot\sum_f \exp\left[j\frac{2\pi(s-n)}{h}f\right].$$

It is easy to see that the sum over f in right hand side is nonzero only for $s=n+l\cdot h$, where $l=0,\pm1,\pm2,...$ Increasing the number of macroparticles h to an infinity we can make terms with $l\neq0$ arbitrarily small because of decreasing of A_{ms}. Note that final equations do not depend on h as it'll be shown below. Therefore it is possible to go to the limit $h\to\infty$. This limit means a transition from discrete macroparticles to a continuous (hollow) beam. Coming from such considerations we'll remain in the sum over s only term with $s=n$:

$$\sum_f \exp\left[\frac{j(m-\xi)\,z_f}{R}\right]\cdot\exp\left[-\frac{2\pi n}{h}f\right]=h\cdot A_{m-\xi,n}(J_z)\cdot e^{jn\Psi_z}.$$

Analogously

$$\sum_{f'} \exp\left[-\frac{j(m-\xi)z_{f'}}{R}\right]\cdot\exp\left[\frac{2\pi n'}{h}f'\right]=h\cdot A_{-(m-\xi),n'}\cdot(J_z)\cdot e^{-jn'\Psi_z}.$$

Substituting these relations into the equations (21) and carrying over the multiplier $e^{-jn\Psi_z}$ into the left hand side we obtain

$$\dot{u}_n e^{-jn\Psi_z}=-\frac{1}{2}e^2\cdot N\cdot v\cdot\frac{\omega_0 R}{m_s\Omega}\cdot\sum_{n'}u_{n'}\sum_{k,m}A_{m-\xi,n}\cdot(J_z)\cdot A_{-(m-\xi),n'}\cdot(J_z)\times$$

$$\times\frac{\partial E_{kz,m}}{\partial x}\cdot\frac{\partial E_{kz,-m}}{\partial x}\cdot\frac{Z_k(-jm\omega_0+j\Omega)}{-m\omega_0+j\Omega}. \qquad (24)$$

And, finally, let's introduce new variables

$$v_n=u_n\cdot e^{-jn\Psi_z}.$$

Differentiating with respect to the time yields

$$\dot{v}_n = \frac{d}{dt}\left(u_n \cdot e^{-jn\psi_z} \right) = \dot{u}_n\, e^{-jn\psi_z} - u_{n\cdot} \cdot e^{-jn\psi_z} \cdot jn\frac{d\psi_z}{dt}\ .$$

As $\qquad \dfrac{d\psi_z}{dt} = \Omega_z,$

then $\qquad \dot{u}_n \cdot e^{-jn\psi_z} = \dot{v}_n + jn\Omega_z v_n .$

Substituting this into (24) and carrying over the second term into right hand side we obtain the set of differential equations, this time without time varying coefficients

$$\dot{v}_n = -\frac{1}{2} e^2 \cdot N \cdot v \cdot \frac{\omega_0 R}{m_s \Omega} \cdot \sum_{n'} v_{n'} \sum_{k,m} A_{m-\xi,n}(J_z)\, A_{-(m-\xi),-n'} \cdot (J_z) \times$$

$$\times \frac{\partial E_{kz,m}}{\partial x} \cdot \frac{\partial E_{kz,-m}}{\partial x} \cdot \frac{Z_k(-jm\omega_0 + j\Omega)}{-jm\omega_0 + j\Omega} - jn\Omega_z v_n . \tag{25}$$

Note that from definition of A_{mn} follows that

$$A_{-(m-\xi),-n'}(J_z) = A_{m-\xi,n'}(J_z)\ .$$

Taking also into account that

$$\frac{e}{m_s} = \frac{c^2}{\gamma \cdot U_0} = \frac{\omega_0^2 R^2}{\gamma \cdot U_0}, \qquad \frac{\Omega}{\omega_0} = \nu, \qquad e \cdot N \cdot f_0 = \frac{e \cdot N \cdot \omega_0}{2\pi} = I_0,$$

where U_0 is a rest energy of particles (in volts), I_0 is an average circulating current in a storage ring, ν is vertical betatron tune, γ is relativistic factor, the set of equations can be rewritten as follows:

$$\dot{v}_n = -\frac{\omega_0 \cdot I_0 \cdot R}{4\pi \cdot \nu \cdot \gamma \cdot U_0} \cdot (2\pi R)^2 \cdot c \sum_{n'} v_{n'} \sum_{k,m} J_n\left(\frac{m-\xi}{R}\sigma_z\right) J_{n'}\left(\frac{m-\xi}{R}\sigma_z\right) \times$$

$$\times \frac{\partial E_{kz,m}}{\partial x} \cdot \frac{\partial E_{kz,-m}}{\partial x} \cdot \frac{Z_k(-jm\omega_0 + j\Omega)}{-jm\omega_0 + j\Omega} - jn\Omega_z v_n . \tag{26}$$

Here σ_z is the bunch length

$$\sigma_z = \sqrt{\frac{2J_z}{M \cdot \Omega_z}}\ .$$

The set of equations (26) can be written briefly in the form

$$\dot{v}_n = \sum_{n'} v_{n'} \cdot M_{nn'} - jn\Omega_z \cdot v_n, \qquad (27)$$

where coefficients $M_{nn'}$ are determined as

$$M_{nn''} = -\frac{\omega_0 \cdot I_0 \cdot R}{4\pi \cdot \nu \cdot \gamma \cdot U_0} (2\pi R)^2 \cdot c \cdot \sum_{k,m} J_n\left(\frac{m-\xi}{R}\sigma_z\right) \cdot J_{n'}\left(\frac{m-\xi}{R}\sigma_z\right) \times$$
$$\times \frac{\partial E_{kz,m}}{\partial x} \cdot \frac{\partial E_{kz,-m}}{\partial x} \cdot \frac{Z_k(-jm\omega_0 + j\Omega)}{-jm\omega_{00} + j\Omega} \quad . \qquad (28)$$

Coefficients $M_{nn'}$ can be expressed via transverse impedance Z_T

$$M_{nn'} = -\frac{\omega_0 \cdot I_0 \cdot \beta_{av}}{4\pi \cdot \nu \cdot \gamma \cdot U_0} \sum_m J_n\left(\frac{m-\xi}{R}\sigma_z\right) \cdot J_{n'}\left(\frac{m-\xi}{R}\sigma_z\right) \cdot Z_T(-jm\omega_0 + j\Omega), \qquad (29)$$

where

$$Z_T(-jm\omega_0 + j\Omega) = c \sum_k (2\pi R)^2 \cdot \left|\frac{\partial E_{kz,m}}{\partial x}\right|^2 \cdot \frac{Z_k(-jm\omega_0 + j\Omega)}{-m\omega_0 + \Omega}$$

and $\beta_{av} = \frac{R}{\nu}$ is an average β‑function.

The expression (29) is valid for narrow and wide band transverse impedances, i.e. for multi‑and single‑turn effects.

If only single turn effects remain valid the sum in matrix elements may be replaced by an integral over *m*:

$$M_{nn'} = -\frac{\omega_0 \cdot I_0 \cdot \beta_{av}}{4\pi \cdot \gamma \cdot U_0} \cdot \int_{-\infty}^{\infty} n\left(\frac{m-\xi}{R}\sigma_z\right) \cdot J_{n'}\left(\frac{m-\xi}{R}\sigma_z\right) \times \qquad (30)$$
$$\times Z_T[-j(m-\nu)\omega_0] \cdot dm.$$

At zero chromaticity (κ = 0) this reduces to

$$M_{nn'} = -\frac{\omega_0 I_0 \beta_{av}}{4\pi\gamma U_0} \int_{-\infty}^{\infty} J_n\left(\frac{m-\nu}{R}\sigma_z\right) J_{n'}\left(\frac{m-\nu}{R}\sigma_z\right) Z_T[-j(m-\nu)\omega_0]\, dm$$

and going to new variables $(m-\nu) \to m$ one gets

$$M_{nn'} = -\frac{\omega_0 \cdot I_0 \cdot \beta_{av}}{4\pi \cdot \gamma \cdot U_0} \cdot \int_{-\infty}^{\infty} J_n\left(\frac{m}{R}\sigma_z\right) \cdot J_{n'}\left(\frac{m}{R}\sigma_z\right) \cdot Z_T[-jm\omega_0] \cdot dm\,. \qquad (31)$$

Note that real part of a transverse impedance is an odd function of frequency and its imaginary part is an even one. In addition, real part of a transverse impedance is positive only for positive frequency in contrast to a longitudinal impedance.

If $n + n'$ is even the product of Bessel function in (31) is even function with respect to m also. Therefore only integral of imaginary part Z_T is nonzero and the corresponding matrix element is pure imaginary. On the contrary, if $n + n'$ is odd then $M_{nn'}$ is pure real.

At nonzero chromaticity ($\kappa \neq 0$) elements of matrix are

$$M_{nn'} = -\frac{\omega_0 I_0 \beta_{av}}{4\pi\gamma U_0} \int_{-\infty}^{\infty} J_n\left(\frac{m-\kappa}{R}\sigma_z\right) \cdot J_{n'}\left(\frac{m-\kappa}{R}\sigma_z\right) \cdot Z_T[-jm\omega_0] \cdot dm \, . \quad (32)$$

As the product of Bessels in the integrand of (32) is nonsymmetric with respect to zero frequency then $M_{nn'}$'s are complex in this case.

With the set of equations (27) the problem of stability can be reduced to the eigen value problem for the matrix $B_{nn'}$:

$$B_{nn'} = M_{nn'} - j\delta_{nn'} \cdot n \cdot \Omega_z ,$$

i.e. to solving characteristic equation with respect to coherent frequency shift $\Delta\omega$:

$$Det[M_{nn'} - j\delta_{nn'} \cdot (n\Omega_z + \Delta\omega)] = 0 \, . \quad (33)$$

Matrix B is twice infinite and for real solving of (33) must be truncated. Note that truncation should be symmetric with respect to zero mode to give correct result.

As the simplest case we'll consider the case of only three modes: zero and ± 1'st with zero chromaticity.

Interaction matrix $M_{nn'}$ is symmetric: $M_{nn'} = M_{n'n}$. Change of sign of numbers n or n' changes or does not change the sign of matrix element $M_{nn'}$ depending on whether this number is odd or even.

Taking these properties into account characteristic equation (33) for three modes can be written as follows

$$\begin{vmatrix} M_{11} + j\Omega_z - j\Delta\omega & -M_{10} & -M_{11} \\ -M_{10} & M_{00} - j\Delta\omega & M_{10} \\ -M_{11} & M_{10} & M_{11} - j\Omega_z - j\Delta\omega \end{vmatrix} = 0 \quad (34)$$

or, after calculation of determinant,

$$(\Delta\omega)^3 + j(2M_{11} + M_{00})(\Delta\omega)^2 - (\Omega_z^2 + 2M_{11} \cdot M_{00} - 2M_{10}^2)\,\Delta\omega - jM_{00}\,\Omega_z^2 = 0 \, .$$

All coefficients of this equation are real because M_{00} and M_{11} at zero chromaticity are pure imaginary and M_{10} is real.

For analysis it is convenient to normalize this equation dividing it by Ω_z^3, thus going to dimensionless variables:

$$\lambda^3 + k\cdot\lambda^2 - (1-d^2)\cdot\lambda - k = 0 \quad , \qquad (35)$$

where

$$\lambda = \Delta\omega/\Omega_z, \quad k = jM_{00}/\Omega_z, \quad d^2 = (2M_{10}^2 - 2M_{11}M_{00})/\Omega_z.$$

Note that we have neglected the term $2M_{11}$ in a coefficient at λ^2 assuming it to be little as compared to M_{00}.

We have obtained cubic equation with real coefficients. If all three roots of this equation are real then frequency shift is real and transverse motion is stable. In another case two roots are complex conjugated and one of modes is unstable. From algebra it is known that there exist a discriminant, i.e. an expression that allows to judge what is the case. For equation obtained, discriminant is

$$D(k,d) = \left(\frac{k^3}{27} - k\cdot\frac{d^2-1}{6} - \frac{k}{2}\right)^2 + \left(\frac{d^2-1}{3} - \frac{k^2}{9}\right)^3.$$

If $D(k,d) < 0$ then we have the case of three real roots and motion is stable. In opposite case two roots are complex conjugated and one of the modes is unstable.

Application of the discriminant is illustrated by Fig.1. Here a curve $D(k,d) = 0$ is represented in coordinates k, d. This curve is a boundary between stable and unstable regions of variables *k,d.* Regions 1 and 3 are stable and region 2 is a region of instability. Note that discriminant is even function of *k.* In Fig.1 a curve is represented for positive *k* only.

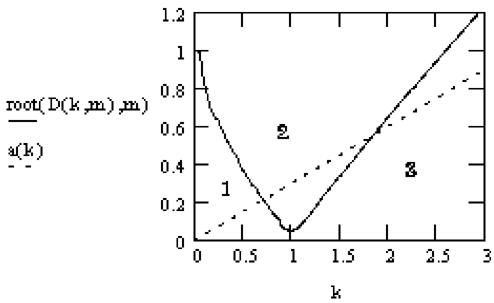

Fig. 1. Stability regions. Straight line: $a(k) = b\cdot k,\ b = 0.3.$

As k and d are both proportional to beam current I_0 their ratio remains constant with current increasing. This relation is represented in Fig.1 by straight line coming from the origin of coordinates. Its intersection point with the curve $D(k,d)=0$ gives the threshold values of k and d. There can exist one or two intersection points depending on a slope of the straight line, i.e. on the ratio of d/k. But second point is unattainable really because of instability that occurs during beam accumulation.

In previous papers on the theory of a feedback [7–9] it was assumed that feedback acts only on the center of gravity of a bunch. Such feedback changes only the value of k (in our notation). If a feedback is reactive a change of k is real. The best case is if the k is reduced by the feedback to zero. Then discriminant $D(k,d)$ gives the condition of stability

$$d^2 - 1 < 0 \qquad \text{or} \qquad 2M_{10}^2 < \Omega_z^2 .$$

Thus, reactive feedback allows one to increase beam current but only to some threshold. Really an increase is not more than 2 – 2.5 times.

Another case was called by R.Ruth [7,8] «an abnormal» regime of the feedback when the value of k is increased by reactive feedback much more than unity. In three mode approach «abnormal» feedback permits one to increase beam current with no limit; really it can be limited by instability of higher modes. But entering the «abnormal» regime is difficult because of necessity to cross unstable region (see Fig.1) during beam accumulation.

It is necessary, however, to emphasize that a reactive feedback acting only on the center of gravity must be pure reactive. The very small resistive component makes some modes unstable with a threshold determined only by radiation damping. So the phase of reactive feedback should be stable with high accuracy. This was found, in particular, by G.Sabbi [14] in his numerical simulations of feedback.

If chromaticity is nonzero the coefficients in characteristic equation become complex, and its roots are complex also. This leads to an instability of one mode. It is nothing but usual head–tail effect with threshold determined by radiation damping.

But a combined application of chromaticity and feedback gives good results in suppression of instability and permits one to increase the threshold of TMCI significantly.

3. Mode equations with feedback

Here we'll derive equations of mode evolution in the presence of feedback. In a storage ring a feedback kicker is assumed to be placed at azimuth $\theta=\theta_0$, electromagnetic field of which is acting on a bunch in transverse direction. The field in a kicker is determined by a voltage from the output of a feedback amplifier. At the input of amplifier a signal acts obtained from the strip line pick-up. Pick-up is located at the azimuth θ = 0. The beam rotates in counterclockwise direction.

The motion of the f-th particle obeys the same equation (12), which was derived in Section 2:

$$\dot{y}_f = \frac{e}{m_s \cdot \Omega} \cdot \overline{e^{-j(\Omega_0 t + \xi \cdot z_f / R)} (E_x - vB_y)}_f , \quad (36)$$

but the electromagnetic force $e(E_x - vB_y)$ should include the field created by the feedback kicker.

This force appears due to electromagnetic field in the kicker strip line:

$$E_x = \frac{V_{kick}}{a} \cdot e^{\gamma(l - L_0)}, \quad vB_y = -\frac{v\mu_0 E_x}{Z_k} = -\frac{v \cdot \mu_0 \cdot V_{kick}}{a \cdot Z_k} \cdot e^{\gamma(l - L_0)},$$

where V_{kick} is the voltage at the input of a kicker, l is a coordinate along a ring, $\gamma = s/c$, a is the gap between deflecting plates of a kicker, L_0 is the distance between pick-up and kicker (kicker is assumed to be the match loaded and connected in the direction opposite to the beam revolution), $Z_k = \sqrt{\mu/\varepsilon}$ (=120 Ohm for vacuum).

After summing up one gets

$$F(l,s) = e \cdot (E_x - vB_y) = e \cdot \frac{V_{kick}}{a} \cdot \exp[\gamma(l - L_0)] \cdot \left(1 + \frac{v \cdot \mu_0}{Z_k}\right).$$

Here v is the particle velocity, $\frac{v \cdot \mu_0}{Z_k} = \frac{v}{c} = \beta_p$.

In equation (36) the Lorentz force should be expressed in coordinates of the accompanying system of an equilibrium particles. For this, first, let's expand $F(l,s)$, which is periodic in l, into Fourier series:

$$F(l,s) = \sum_m F_m(s) \cdot \exp\left(j \frac{ml}{R}\right),$$

where $$F_m(s) = \frac{1}{2\pi R} \int_{L_0 - L_2}^{L_0} F(l,s) \cdot \exp\left(-j \frac{ml}{R}\right) \cdot dl .$$

Here L_2 is the length of the kicker. Substituting $F(l,s)$ yields

$$F_m(s) = e \cdot \frac{V_{kick}}{a} \cdot (1+\beta) \cdot \frac{1}{2\pi R} \int\limits_{L_0 - L_2}^{L_0} \exp[\gamma(l - L_0)] \cdot \exp\left(-j\frac{ml}{R}\right) \cdot dl =$$

$$= e \cdot \frac{V_{kick}}{a} \cdot (1+\beta) \cdot \frac{1}{2\pi R} \cdot \frac{\exp(-j\frac{mL_0}{R})}{\gamma - jm/R} \cdot \left\{1 - \exp\left[-\left(\gamma - j\frac{m}{R}\right)L_2\right]\right\}$$

and Fourier series takes the form:

$$F(l,s) = \frac{1}{2\pi R} \cdot e \cdot \frac{V_{kick}}{a} \cdot (1+\beta) \cdot \sum_m \exp\left(j\frac{ml}{R}\right) \cdot \frac{\exp(-j\frac{mL_0}{R})}{\gamma - jm/R} \times$$

$$\times \{1 - \exp\left[-\left(\gamma - j\frac{m}{R}\right)L_2\right].$$

For transition to accompanying coordinate system one must change variables $l = z + \omega_0 Rt$ (*z* being longitudinal coordinate in this system), beforehand passing to the time domain by the reverse Laplace transformation

$$F(l,t) = \frac{e}{2\pi R} \cdot \frac{1+\beta}{a} \cdot \sum_m \exp\left[j\frac{m(l - L_0)}{R}\right] \cdot L^{-1}\left\{V_{kick}(s) \cdot \frac{1 - \exp[-(\gamma - j\frac{m}{R})L_2]}{\gamma - j\frac{m}{R}}\right\}.$$ N

ow let's change l by $z + \omega_0 R$

$$F(z,t) = \frac{e}{2\pi R} \cdot \frac{1+\beta}{a} \cdot \sum_m \exp\left(j\frac{mz}{R}\right) \cdot \exp\left(-j\frac{mL_0}{R}\right) \times$$

$$\times L^{-1}\left\{V_{kick}(s - jm\omega_0) \cdot \frac{1 - \exp[-(\gamma_m - j\frac{m}{R})L_2]}{\gamma_m - j\frac{m}{R}}\right\}.$$

Here

$$\gamma_m = \frac{s - jm\omega_0}{\beta \cdot c}.$$

Note that direct and reverse Laplace transformation are fulfilled over fast time, i.e. over betatron oscillation time. Slowly changing coordinate z at this transformation is considered as a constant.

The force $F(z,t)$ is calculated above for arbitrary point z. But we are interested of this force at the point z_f where the particle with number f is situated:

$$F(z_f,t) = \frac{e}{2\pi R}\cdot\frac{1+\beta}{a}\cdot\sum_m \exp\left[j\frac{mz_f}{R}\right]\cdot\exp\left[-j\frac{mL_0}{R}\right]\times$$

$$\times L^{-1}\left\{V_{kick}(s-jm\omega_0)\cdot\frac{1-\exp[-(\gamma_m - j\frac{m}{R})L_2]}{\gamma_m - j\frac{m}{R}}\right\}. \qquad (37)$$

The voltage at the input of a kicker can be expressed as

$$V_{kick}(s) = K(s)\cdot e^{s\tau}\cdot V_1(s),$$

where V_1 is a voltage at the output of pick-up, $K(s)$ is the gain factor of a feed-back amplifier, τ is the signal time delay in a cable between pick-up and a kicker.

Next we'll calculate the pick-up output voltage $V_1(s)$. For matched strip line pick-up this voltage is

$$V_1(s) = -\frac{Z_2}{2}\cdot(I_{1b} + e^{-\gamma L_1}\cdot I_{2b}), \qquad (38)$$

where Z_0 is characteristic impedance of strip line, L_1 is the length of pick-up line and induced currents are

$$I_{1b}(s) = \int_{A_1}\vec{j}(x,y,l;s)\cdot\vec{e}_1(x,y,l)\cdot dx\cdot dy\cdot dl,$$

$$I_{2b}(s) = \int_{A_2}\vec{j}(x,y,l;s)\cdot\vec{e}_2(x,y,l)\cdot dx\cdot dy\cdot dl.$$

Here $\vec{e}_1$ and $\vec{e}_2$ are the end field of strip line, $\vec{j}$ is the beam current density, A_1 and A_2 are the end regions of the strip line.

In time domain induced current $I_{1b}(t)$ is

$$I_{1b}(t) = \int_{A_1}\vec{j}(x,y,z;t)\cdot\vec{e}_1(x,y,l)\cdot dx\cdot dy\cdot dl, \qquad (39)$$

where $z = l - \omega_0 R\cdot t$.

The current density $\vec{j}(x,y,z;t)$ is periodic in z and can be expanded into Fourier series

$$\vec{j}(x,y,z;t)=\sum_{m=-\infty}^{\infty}\vec{j}_m(x,y;t)\cdot\exp\left(j\frac{mz}{R}\right)=\sum_{m=-\infty}^{\infty}\vec{j}_m(x,y;t)\times$$

$$\times\exp\left(j\frac{ml}{R}\right)\cdot\exp(-jm\omega_0 t)\,. \qquad (40)$$

Here

$$\vec{j}_m(x,y;t)=\frac{1}{2\pi R}\int_{-\pi R}^{\pi R}\vec{j}(x,y,z,t)\cdot\exp\left(-j\frac{mz}{R}\right)\cdot dz.$$

Substituting this into (40) and then into (39) gives

$$I_{1b}(t)=\sum_m\frac{1}{2\pi R}\int_{A_1}\vec{j}(x,y,z;t)\cdot\exp\left(j\frac{mz}{R}\right)\cdot\vec{e}_1(x,y,l)\cdot\exp\left(j\frac{m\cdot l}{R}\right)\times$$

$$\times\exp(-j\omega_0 t)\cdot dx\cdot dy\cdot dl\cdot dz\,.$$

Note that

$$\int_{-\pi R}^{\pi R}\vec{e}_1(x,y,l)\cdot\exp\left(j\frac{m\cdot l}{R}\right)\cdot dl=2\pi R\cdot\vec{e}_{1,-m}(x,y)\,.$$

In this way

$$I_{1b}(t)=\sum_m\int\vec{j}(x,y,z;t)\cdot\exp\left(j\frac{m\cdot z}{R}\right)\vec{e}_{1,-m}(x,y)\cdot\exp(-jm\omega_0 t)\,dx\cdot dy\cdot dz\,. \qquad (41)$$

For the point‑like bunch

$$\vec{j}(x,y,z;t)=e\cdot N\cdot\vec{v}\cdot\delta(x-x_0)\cdot\delta(y)\cdot\delta(z-z_0),$$

where $x_0(t)$, $z_0(t)$ are coordinates of the particle.

Assuming that particle velocity possesses only longitudinal (z‑th) component and substituting current density into (41) we get

$$I_{1b}(t)=e\cdot N\cdot v\cdot\sum_m e_{1z,-m}(x_0)\cdot\exp(-j\frac{m\cdot z_0}{R})\cdot\exp(-jm\omega_0 t)\,. \qquad (42)$$

Analogously

$$I_{2b}(t)=e\cdot N\cdot v\cdot\sum_m e_{2z,-m}(x_0)\cdot\exp(-j\frac{m\cdot z_0}{R})\cdot\exp(-jm\omega_0 t)\,. \qquad (43)$$

The expression (38) for the voltage $V_1(s)$ at the output of pick‑up contains the Laplace transforms of induced currents. The last as it follows from (42) and (43) are

$$I_{1b}(s) = e \cdot N \cdot v \cdot \sum_m L\left\{e_{1z,-m}(x_0) \cdot \exp\left(-j\frac{m \cdot z_0}{R}\right) \cdot \exp(-jm \cdot \omega_0 t)\right\},$$

$$I_{2b}(s) = e \cdot N \cdot v \cdot \sum_m L\left\{e_{2z,-m}(x_0) \cdot \exp\left(j\frac{m \cdot z_0}{R}\right) \cdot \exp(-jm \cdot \omega_0 t)\right\}.$$

These induced currents are written for one particle with coordinates x_0, z_0, which is a source of the field. But in our case we consider several macroparticles in a separatrix (e.g., *h* macroparticles). The full current is then equal to the sum of the currents of all macroparticles in one separatrix (e.g., *h* macroparticles)

$$I_{1b}(s) = \frac{e \cdot N \cdot v}{h} \sum_{f'=1}^{h} \sum_m L\left\{e_{1z,-m}(x_{f'}) \cdot \exp\left(-j\frac{m \cdot z_{f'}}{R}\right) \cdot e^{-jm\omega_0 t}\right\}, \quad (44)$$

$$I_{2b}(s) = \frac{e \cdot N \cdot v}{h} \sum_{f'=1}^{h} \sum_m L\left\{e_{2z,-m}(x_{f'}) \cdot \exp\left(-j\frac{m \cdot z_{f'}}{R}\right) \cdot e^{-jm\omega_0 t}\right\} \quad (45)$$

The next issue to be considered are functions $e_{1z,-m}(x_{f'})$ and $e_{2z,-m}(x_{f'})$. For further calculations it is convenient to locate the origin of longitudinal coordinate in laboratory system (l) at the beginning of the pick-up strip line. The simplest case is to assume these functions in the form of δ-functions

$$e_{1z}(x,l) = E(x) \cdot \delta(l); \qquad e_{2z}(x,l) = -E(x) \cdot \delta(l - L_1).$$

Normalization should be such that e_{1z} and e_{2z} were the field values at the unity potential difference between the plates of strip line. Then

$$e_{1z,m} = \frac{E(x)}{2\pi R}; \qquad e_{2z,m} = -\frac{E(x)}{2\pi R} \cdot \exp\{-j\frac{m \cdot L_1}{R}).$$

In some vicinity of $x = 0$ the function $E(x)$ can be presented approximately as

$$E(x) \cong E'(0) \cdot x.$$

Substituting this into (44) and (45) one gets

$$I_{1b}(s) = \frac{e \cdot N \cdot v}{h} \sum_{f'=1}^{h} \sum_m \frac{E'(0)}{2\pi R} \cdot L\{x_{f'} \cdot \exp(-jm\omega_0 t\} \cdot \exp\left(-j\frac{m \cdot z_{f'}}{R}\right). \quad (46)$$

$$I_{2b}(s) = -\frac{e \cdot N \cdot v}{h} \sum_{f'=1}^{h} \sum_m \frac{E'(0)}{2\pi R} \cdot L\{x_{f'} \cdot \exp(-jm\omega_0 t\} \cdot \exp\left(-j\frac{m \cdot z_{f'}}{R}\right) \times$$

$$\times \exp\left(-j\frac{m \cdot L_1}{R}\right). \quad (47)$$

Output signal voltage then will be

$$V_1(s) = -\frac{Z_0}{2}(I_{1b} + e^{-\gamma L_1} \cdot I_{2b}) = -\frac{Z_0}{2} \cdot \frac{eN\nu}{h} \cdot \sum_{f'=1}^{h} \sum_m \frac{E'(0)}{2\pi R} \times$$

$$\times L\{x_{f'} \cdot e^{-jm\omega_0 t}\} \cdot \exp\left(-j\frac{m \cdot z_{f'}}{R}\right)\left\{1 - \exp\left[\left(j\frac{m}{R} - \gamma\right) \cdot L_1\right]\right\}.$$

As a result we obtain

$$V_{kick}(s) = K(s) \cdot e^{-s\tau} \cdot V_1(s) = -K(s) \cdot e^{-s\tau} \cdot \frac{Z_0}{2} \cdot \frac{eN\nu}{h} \cdot \sum_{f'=1}^{h} \sum_n \frac{E'(0)}{2\pi R} \times$$

$$\times L\{x_{f'} \cdot e^{-jn\omega_0 t}\} \cdot \exp\left(-j\frac{nz_{f'}}{R}\right) \cdot \left\{1 - \exp\left[\left(j\frac{n}{R} - \gamma\right)L_1\right]\right\}. \qquad (48)$$

Here index of summation m is changed by n to distinguish from that in sum for $F(z_f, t)$ (see (37)).

Now the expression (48) for $V_{kick}(s)$ should be substituted into $F(z_f, s)$ (see (37)) and then into equation of motion (36). After averaging by method described in Section 2 we'll get the set of equation for time evolution of complex amplitudes of transverse oscillations of the macroparticles in the presence of a feedback

$$\dot{y}_f = -\frac{Z_0}{2j} \cdot \frac{eN\nu}{h} \cdot \frac{e}{m_s\Omega} \cdot \frac{1+\beta}{(2\pi R)^2} \cdot \frac{E'(0)}{a} \cdot \sum_{f'=1}^{h} y_{f'} \cdot \sum_m K(-jm\omega_0 + j\Omega) \times$$

$$\times e^{(jm\omega_0 - j\Omega)\tau} \cdot e^{-jmL_0/R} \cdot \left[1 - \exp\left(j\frac{mL_1}{R}\right) \cdot \exp(-\gamma_m \cdot L_1/R)\right] \times \qquad (49)$$

$$\times \frac{1 - \exp(\gamma_m - jm/R) \cdot L_2}{\gamma_m - jm/R} \cdot e^{j(m-\xi)z_f/R} \cdot e^{-j(m-\xi)z_{f'}/R}.$$

Written above equation set (49) does not contain yet the terms corresponding to the wake field forces. Its transformation with going to the mode variables as it was done in Section 2 and adding the wake field terms yields

$$\dot{v}_n = -\sum_{n'} v_{n'} . (M_{nn'} + F_{nn'}) - jn\Omega_z v_n, \qquad (50)$$

where

$$M_{nn'} = \frac{\omega_0 I_0 \beta_{av}}{4\pi \cdot \gamma \cdot U_0} \int_{-\infty}^{\infty} J_n\left(\frac{m-\xi}{R} \cdot \sigma\right) \cdot J_{n'}\left(\frac{m-\xi}{R} \cdot \sigma\right) \cdot Z_T[-j(m-\nu)\omega_0] \cdot dm \quad (51)$$

and

$$F_{nn'} = \frac{\omega_0 I_0 \beta_{av}}{4\pi \cdot \gamma \cdot U_0} \cdot \frac{Z_0 E'(0) \cdot R}{2a} \sum_{m=-\infty}^{\infty} J_n\left(\frac{m-\xi}{R} \cdot \sigma\right) \cdot J_{n'}\left(\frac{m-\xi}{R} \cdot \sigma\right) \times$$
$$\times K[-j(m-\nu)\omega_0] \cdot e^{j(m-\nu)\omega_0 \tau} \cdot e^{jmL_0 / R} \times \quad (52)$$
$$\times \cdot \frac{\{1-\exp[-(\gamma_m - jm/R)L_1]\} \cdot \{1-\exp[-(\gamma_m - jm/R)L_2]\}}{m - \frac{\nu}{2}}.$$

Here $\gamma_m - jm/R = j(\nu - 2m)/R$, $\beta_p = 1$.

Let's analyze expression (52) for $F_{nn'}$. There are rapidly oscillating factors under the sum symbol, namely, exponents with *m* index in their power:

$$e^{jm\omega_0 \tau} \cdot e^{-jmL_0 / R} = e^{jm(\omega_0 \tau - L_0 / R)}.$$

But these exponents vanish if the relation is satisfied

$$\omega_0 \tau - L_0 / R = 0. \quad (53)$$

This relation determines the relevant delay time in a feedback cable

$$\tau = \frac{L_0}{\omega_0 \cdot R} = \frac{L_o}{c}.$$

This delay is the duration of the particle travel from pick-up to kicker. Under this condition a bunch and a feedback signal arrive to the kicker simultaneously.

Character of the feedback (reactive or resistive) is determined by the exponent remaining in $F_{nn'}$:

$$e^{-j\nu\omega_0 \tau} = e^{-j\nu L_0 / R},$$

in other words, by the distance between pick-up and a kicker. But this distance cannot be adjusted operatively. The way to control a character of a feedback is, as it is known, to use two pick-ups separated by a quarter of a betatron wavelength.

The signals from pick-ups should be summed with arbitrarily regulated scale coefficients k_1 and k_2:

$$e^{-j\nu L_0 / R} \cdot (k_1 + j \cdot k_2).$$

Adjusting of coefficients k_1 and k_2 allows one to have reactive or resistive or some intermediate feedback.

The expression

$$\frac{Z_0 \cdot E'(0) \cdot R}{2a} \cdot K[-j(m-\nu)\omega_0] \frac{(1-e^{-(\gamma_m - jm/R)L_1})(1-e^{-(\gamma_m - jm/R)L_2})}{m-\frac{\nu}{2}}$$

contained in $F_{nn'}$ have a dimension of transverse impedance (*Ohm/m*). We denote it as the transverse feedback impedance $Z_F[-j(m-\nu)\omega_0]$. Then

$$F_{nn'} = \frac{\omega_0 \cdot I_0 \cdot \beta_{av}}{4\pi \cdot \gamma \cdot U_0} \cdot e^{-j\nu L_0/R} \cdot (k_1 + j \cdot k_2) \times$$

$$\times \sum_{m=-1}^{\infty} J_n\left(\frac{m-\xi}{R} \cdot \sigma_z\right) \cdot J_{n'}\left(\frac{m-\xi}{R}\sigma_z\right) \cdot Z_F[-j(m-\xi)\omega_0]\,. \qquad (54)$$

This expression is valid only if the time delay of feedback signal is equal to particles travel time (see (53)). Determined by equation (53) delay may be too large for large accelerators, of the order of tens to hundred microseconds. In real cable with so long delay there would be very large attenuation of the feedback signal. Digital device can create such delay but this device can reproduce only signal proportional to the coordinate of center of gravity of a bunch but not to higher moments of a bunch. Therefore in this case only F_{00} would be nonzero.

Frequency response of a feedback impedance is determined by that of the amplifier and the length of pick-up and kicker lines. If the band-width of the feedback includes many harmonics (m's) only the single-turn effects are significant and the sum by m can be changed by an integral over m:

$$F_{00} = \frac{\omega_0 \cdot I_0 \cdot \beta_{av}}{4\pi \cdot \gamma \cdot U_0} e^{-j\nu L_0/R}(k_1 + j \cdot k_2) \int_{-\infty}^{\infty} J_0^2\left(\frac{m-\xi}{R}\sigma_z\right) Z_F[-j(m-\nu)\omega_0]\,dm\,.$$

Thus, by variation of k_1 and k_2 a feedback can be made with arbitrary phase. Its gain can be regulated by value of Z_F (e.g. by the gain of amplifier). But only F_{00} is nonzero.

4. Stability analysis with feedback

Stability analysis is brought now to eigenvalue problem for the set of differential equation (50). But we'll reduce, at first, this set to dimensionless form to facilitate its analysis. A solution of the set can be written in the form:

$$v_n = V_n \cdot \exp(j\Delta\omega \cdot t)\ ,$$

where V_n is an amplitude, $\Delta\omega$ is a frequency shift of the oscillation mode. Imaginary part of the frequency shift gives decrement or increment of corresponding oscillation mode. Positive imaginary part corresponds to damping of the oscillation mode and negative one corresponds to antidamping, i.e., to instability of oscillation.

Substitution the solution into equation set yields

$$j\Delta\omega\cdot V_n = -\sum_{n'} V_{n'}\cdot(M_{nn'} + F_{nn'}) - jn\Omega_z\cdot V_n\ . \qquad (55)$$

Further, let us divide this equation set by synchrotron frequency Ω_z , denoting $\Delta\omega/\Omega_z = \lambda$; $M_{nn'}/\Omega_z = m_{nn'}$; $F_{nn'}/\Omega_z = f_{nn'}$:

$$j\lambda\cdot V_n = \sum_{n'} V_{n'}\cdot(m_{nn'} + f_{nn'}) - jn\cdot V_n$$

or

$$\sum_{n'} V_{n'}\cdot(m_{nn'} + f_{nn'}) + j(n+\lambda)\cdot V_n = 0\ . \qquad (56)$$

Equation set (56) can be rewritten in a form

$$\sum_{n} V_{n'}\cdot(m_{nn'} + f_{nn'} + jn\cdot\delta_{nn'}) + j\lambda\cdot V_n = 0\ . \qquad (57)$$

It is now the set of a uniform linear algebraic equations that has nontrivial solution for values of $j\lambda$ that are minus eigenvalues of matrix *B*, which elements are

$$B_{nn'} = m_{nn'} + f_{nn'} + jn\cdot\delta_{nn'}. \qquad (58)$$

Now, values $m_{nn'}$ are

$$m_{nn'} = \frac{\omega_0\cdot I_0\cdot\beta_{av}\cdot R}{4\pi\cdot\gamma\cdot U_0\cdot\Omega_z}\cdot\int_{-\infty}^{\infty} J_n\left(\frac{m-\xi}{R}\cdot\sigma_z\right)\cdot J_{n'}\left(\frac{m-\xi}{R}\cdot\sigma_z\right)\times$$
$$\times Z_T[-j(m-\nu)\omega_0]\cdot dm\ .$$

Let us change variables in integral (keeping in mind that $\xi = \nu - \kappa$)

$$(m-\nu)\to m;\qquad m = x\cdot\frac{R}{\sigma_z};\qquad y = \kappa\cdot\frac{\sigma_z}{R}$$

then $m_{nn'}$ are

$$m_{nn'} = \frac{\omega_0\cdot I_0\cdot\beta_{av}\cdot R}{4\pi\cdot\gamma\cdot U_0\cdot\sigma_z\cdot\Omega_z}\int_{-\infty}^{\infty} J_n(x+y)\cdot J_{n'}(x+y)\cdot Z_T\left(-j\frac{R\cdot\omega_0}{\sigma_z}x\right)\cdot dx\ . \qquad (59)$$

Further, for Z_T we take the form of that for broad band resonator impedance with resonant frequency ω_c and quality factor Q:

$$Z_T(-jm\omega_0) = \frac{R_T}{-\frac{m\omega_0}{\omega_c} + jQ \cdot \left(\frac{m^2 \omega_0^2}{\omega_c^2} - 1 \right)} \quad ,$$

or, substituting $m = x \cdot \frac{R}{\sigma_z}$; and $p = \frac{R}{\sigma_z} \cdot \frac{\omega_0}{\omega_c}$, we get

$$Z_T(-jm\omega_0) = \frac{R_T}{-p \cdot x + jQ \cdot (p^2 x^2 - 1)} = R_T \cdot z_T(-px). \qquad (60)$$

For $m_{nn'}$ we write now

$$m_{nn'} = \frac{\omega_0 \cdot I_0 \cdot \beta_{av} \cdot R \cdot R_T}{4\pi \cdot \gamma \cdot U_0 \cdot \sigma_z \cdot \Omega_z} \cdot \int_{-\infty}^{\infty} J_n(x+y) \cdot J_{n'}(x+y) \cdot z_T(-px) \cdot dx \,. \qquad (61)$$

We'll represent average current I_0 for convenience of analysis in a form

$$I_0 = I_b \cdot I,$$

where I is dimensionless current and I_b is certain characteristic current so determined that coefficient in front of integral in (61) would be the unity

$$\frac{\omega_0 \cdot I_b \cdot \beta_{av} \cdot R \cdot R_T}{4\pi \cdot \gamma \cdot U_0 \cdot \sigma_z \cdot \Omega_z} = 1 \,.$$

The current I_b is determined by this equation

$$I_b = \frac{4\pi \cdot \gamma \cdot U_0 \cdot \sigma_z \cdot \nu_s}{\beta_{av} \cdot R_T \cdot R} \,. \qquad (62)$$

Now element $m_{nn'}$ is

$$m_{nn'} = I \cdot i_{nn'} = I \cdot \int_{-\infty}^{\infty} J_n(x+y) \cdot J_{n'}(x+y) \cdot z_T(-px) \cdot dx \,. \qquad (63)$$

Note that y is present in the integrand due to chromaticity . For zero chromaticity it is absent and integral is pure real or imaginary depending on whether the sum $n + n'$ is even or odd.

Analogously, for only nonzero f_{00} we get

$$f_{00} = I \cdot j_{00} = I \cdot \frac{R_F}{R_T} \cdot e^{-j\nu L_0 / R} \cdot (k_1 + j \cdot k_2) \cdot \int_{-\infty}^{\infty} J_0^2(x+y) \cdot z_F(-px) \cdot dx \,. \quad (64)$$

It is assumed that value j_{00} can be made with arbitrary phase and module proportional to current I.

Thus, the elements of matrix B are

$$B_{nn'} = I \cdot i_{nn'} + I \cdot j_{nn'} + jn \cdot \delta_{nn'}. \tag{65}$$

The symmetry properties of $i_{nn'}$ are analogous to that of $M_{nn'}$, Namely, $i_{n'n} = i_{nn'}$; change of sign of n or n' changes or does not change the sign of $i_{nn'}$, depending on whether this number is odd or even.

We'll limit our consideration of the stability with feedback to the approach of three modes: –1, 0 +1. Corresponding matrix with the account of symmetry of $i_{nn'}$ is

$$B = \begin{bmatrix} I \cdot i_{11} - j & -I \cdot i_{10} & -I \cdot i_{11} \\ -I \cdot i_{10} & I \cdot (i_{00} + j_{00}) & I \cdot i_{10} \\ -I \cdot i_{11} & I \cdot i_{10} & I \cdot i_{11} + j \end{bmatrix}. \tag{66}$$

The frequency shift $\lambda = \Delta\omega / \Omega_z$ can be found through the eigenvalues b_k of this matrix

$$\lambda_k = \frac{\Delta\omega}{\Omega_z} = jb_k.$$

The k th mode is stable if the imaginary part of λ_k is positive.

Further analysis shows that stability of all three modes can be ensured at negative chromaticity. In this case, as it is well known, the zero mode is unstable and ± 1 modes are damped. Introducing the resistive feedback we can damp the zero mode also, the ± 1 modes remaining stable.

This statement can be verified by given above formulae. We take for illustration data of the former LEP storage ring. These data relate to 90/60 optics, which was used in 1994–96. For this optics there are following relevant data (at injection) [18]: $E = 22$ GeV, $R = 4245$ m, $f_0 = 11.245$ kHz, $Q_s = 0.014817$, σ_z = 1.834 cm, $\alpha = 1.855.10^{-4}$.

Unfortunately, the accurate date concerning broad-band transverse impedance was not found. For definiteness we take $Q_s =$ 1.5, $f_c =$ 1.5 GHz that is not so far from reality. Calculation gives for these data

$$p = \frac{R}{\sigma_z} \cdot \frac{\omega_0}{\omega_c} = 1.735.$$

Below we present plots of real and imaginary parts of λ_k vs. dimensionless current I at p = 1.735, $Q = 1.5$ and various values of y and j_{00}.

Fig.2 represents Re λ_k and Im λ_k vs. I at zero chromaticity y and zero feedback j_{00}. It is a picture of well known TMC instability with the threshold I approximately 1.083.

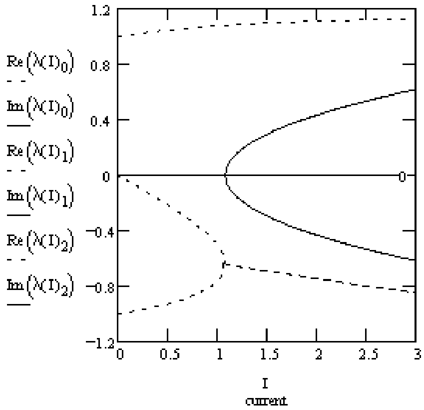

Fig. 2. Plots of Re λ and Im λ vs.I. p = 1.735; Q = 1.5; y = 0; j_{00} = 0.

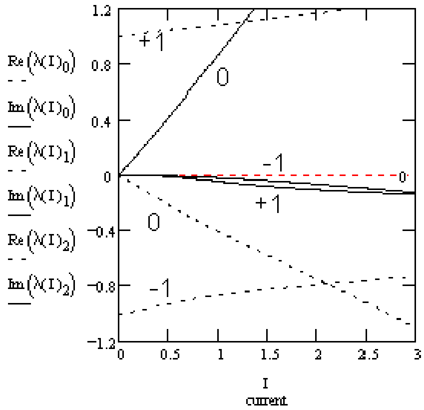

Fig. 3. Re λ and Im λ vs.I. p = 1.735; Q = 1.5; y = 0; j_{00} = 0.8.

In Fig.3 there are depicted the same plots but with resistive feedback. From Fig.3 one can see that at positive j_{00} resistive feedback ± 1 modes are unstable, the zero mode being stable.

In Fig.4 Re λ_κ and Im λ_k vs. I are presented at negative chromaticity but without feedback. As can be seen from this Figure at negative chromaticity zero mode is unstable and ± 1 modes are stable. It is well known head–tail instability at negative chromaticity.

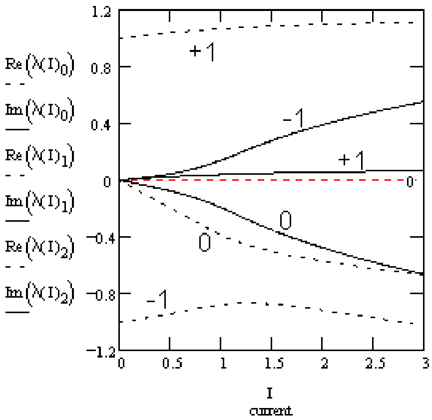

Fig. 4. Re λ and Im λ vs.I. p = 1.735; Q = 1.5; y = – 0.3; j_{00} = 0.

And, finally, Fig.5 and Fig.6 represent Re λ_κ and Im λ_k vs. I at negative chromaticity and with resistive feedback. The difference between Fig.5 and Fig.6 consist in frequency range of a broad band resonator transverse impedance. In Fig.6 frequency range is higher than in Fig.5 (*p* is less because f_c is higher).

As it follows from Fig.5 and Fig.6 combined application of negative chromaticity and resistive feedback makes possible one to keep stable all three modes.

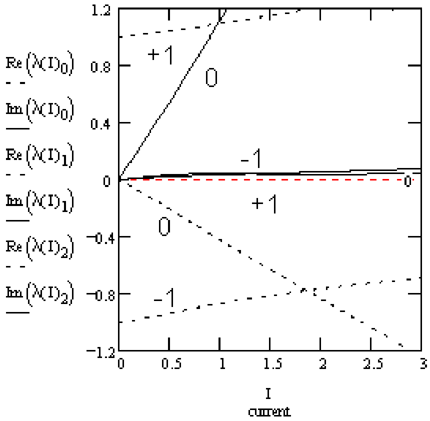

Fig. 5. Re λ and Im λ vs.I. p = 1.735; Q = 1.5; y = –0.3; j_{00} = 1.2.

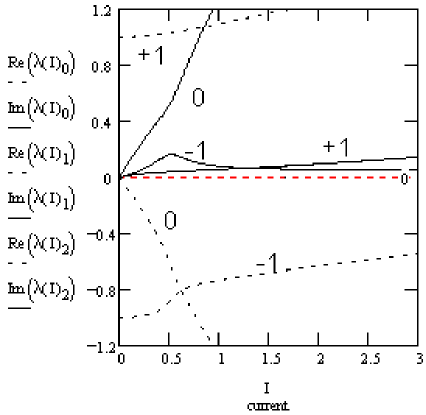

Fig. 6. Re λ and Im λ vs.I. p = 0.7; Q = 1.5; y = □0.7; j_{00} = 1.8.

Now let us estimate a chromaticity necessary for stabilization. In plots Fig.5 the value y is assumed to be –0.3. From definition of y follows that

$$\nu' = \alpha \cdot \kappa = \alpha \cdot y \cdot \frac{R}{\sigma_z}.$$

Using data of LEP yields $\nu' \cong -12.9$. It is not so large chromaticity if one takes into account that natural (uncorrected) chromaticity of LEP was near –200 [19].

These results were obtained in the three mode approach. In this approach a threshold of instability is not seen: the beam is stable at any current. For checking the five mode approach was used. As the five mode approach have shown the threshold appears again but at current 3 – 5 times more than without feedback.

An advantage of described method of stabilization of TMC instability consists in large enough tolerances that takes place for all values involved. It is admissible for a resistive feedback (in combination with negative chromaticity) to acquire some and not small reactive component in contrast to reactive feedback that demands highly tight phase tolerances.

5. Conclusion

In this paper the mathematical formalism have been developed that allows one to analyze Transverse Mode Coupling Instability with feedback in rather simple way. The result is that combined application of negative chromaticity and resistive feedback can increase the threshold of instability by factor of 3 – 5. The large tolerances for parameters of feedback are admissible.

Acknowledgements. I thank my former student (1996) A.A.Krasil'nikov for his contribution to Section 3 of this paper [20].

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