



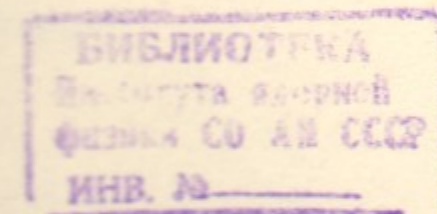
27.19  
1983

35

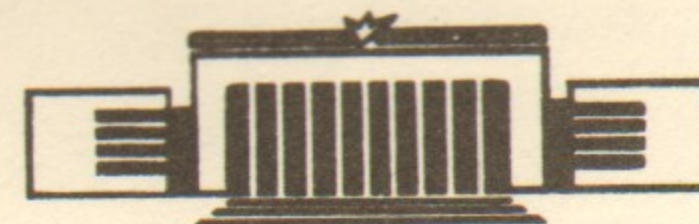
ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ СО АН СССР

V.E.Zakharov and B.G.Konopelchenko

ON THE THEORY OF RECURSION OPERATOR



PREPRINT 83-151



НОВОСИБИРСК

ON THE THEORY OF RECURSION OPERATOR

V.E.Zakharov\* and B.G.Konopelchenko\*\*

\*L.D.Landau Institute of Theoretical  
Physics, Kosigin Str. 2, Moscow,  
117334, USSR

\*\*Institute of Nuclear Physics, 630090,  
Novosibirsk-90, USSR

A b s t r a c t

The general structure and properties of recursion operators for Hamiltonian systems with finite and continual number of the degrees of freedom are considered. Weak and strong recursion operators are introduced. The conditions which determine weak and strong recursion operators are found.

In the theory of nonlinear waves a method for the calculation of recursion operator, which is based on the use of expansion into a power series over the fields and the momentum representation, is proposed. Within the frames of this method a recursion operator is easily calculated via the Hamiltonian of a given equation. It is shown that only the one-dimensional nonlinear evolution equations can possess a regular recursion operator. In particular, the Kadomtsev-Petviashvili equation has no regular recursion operator.

## I. Introduction

The inverse scattering transform method gives a possibility of investigating in detail a wide class of both the ordinary and partial differential equations (see e.g. [1-3]). The equations, integrable by the inverse scattering transform method, possess a number of remarkable properties: solitons, infinite sets of conservation laws, infinite symmetry groups, complete integrability, etc. It turns out also that the equations, to which the inverse scattering transform method is applicable, have the pronounced recursion structure. The so-called recursion operator plays a central role in the formulation of these recursion properties. The role of the recursion operator is two-fold. Firstly, it allows one to write out the families of equations integrable by given spectral problem in a compact form. For example, the family of equations connected with the famous Korteweg-de Vries (KdV) equation can be represented as follows:

$$\frac{\partial u(x,t)}{\partial t} + \partial L^n u = 0 \quad (1.1)$$

where  $\partial \equiv \frac{\partial}{\partial x}$ ,  $n = 0, 1, 2, \dots$ , and the recursion operator  $L$  is

$$L = \partial^2 + 2u + 2\partial^{-1}u\partial \quad (1.2)$$

The KdV equation corresponds to  $n = 1$ . The recursion operator (1.2) for the KdV family of equations was first introduced by Lenart (see [4]).

It follows from (1.1) that the recursion operator allows us to obtain the whole family, starting from one equation (e.g. with  $n = 0$ ). Recursion operators with such a property exist for the other families of equations too: see [5] and subsequent papers on this subject.

The second important role of recursion operators is asso-

ciated with the Hamiltonian treatment of integrable equations. The Hamiltonian structure of equations (1.1) and of the other equations, integrable by the inverse scattering transform method has been investigated, starting from [6,7] in a variety of papers (see [1-3]). It was demonstrated in [8,9] that the integrable equations have a very special structure from the point view of Hamiltonian formalism, namely the whole infinite sets of Hamiltonian structures correspond to these equations. For example, each of equations (1.1) is a Hamiltonian one with respect to the infinite family of Poisson brackets of the form

$$\{F, H\}_n = \int_{-\infty}^{+\infty} dx \frac{\delta F}{\delta u(x)} \partial L^n \frac{\delta H}{\delta u} \quad (1.3)$$

$$n = 0, \pm 1, \pm 2, \dots$$

where the operator  $L$  is given by (1.2). In a similar manner, recursion operators determine the families of Hamiltonian structures for the other integrable equations.

So, a recursion operator is the generating operator for the family of equations connected with a given equation and simultaneously the generating operator for the family of Hamiltonian structures. Combination of these two properties in the same operator indicates the importance of recursion operators in the theory of integrable equations.

An important step in the formulation of the theory of recursion operator was the paper [10]. In this paper it was demonstrated how to calculate the recursion operator for the equations integrable by the second-order matrix spectral problem. The Hamiltonian structure of these equations has been considered in paper [11] in which the remarkable properties of the recursion operator were employed to a considerable extent. The method of calculation of recursion operators, based on the use of the spectral problem, has been further developed in [12-16]. By this method the recursion operator has been calculated for

a wide class of spectral problems [12-23] (see also the paper [24,25]).

The other methods which do not use the spectral problems were suggested in [26-29]. In the papers [26] the recursion operator appears in the Hamiltonian systems which possess the Hamiltonian pairs of operators. In the papers [27-29] some analogs of the recursion operator (hereditary, strong symmetry operators) were considered. In virtue of some postulates, these operators satisfy certain equations. Some solutions of these equations and thereby some examples of the recursion operators have been found. The geometrical structures, connected with the integrable equations and recursion operators, and their properties have been discussed in [30].

In the present paper we consider the different aspects of the theory of recursion operators for Hamiltonian equations. Firstly, we introduce the notions of weak and strong recursion operators. A recursion operator in a "weak" sense (briefly, a weak recursion operator) is the operator which allows us to construct recursively the infinite family of Hamiltonian equations, starting from a given Hamiltonian equation. There exist two types of weak recursion operators. A recursion operator of the first type (H-weak recursion operator) is the operator which converts the gradients of functionals into gradients. A recursion operator of the second type ( $\Omega$  - weak recursion operator) is the operator which transforms symplectic forms into symplectic forms. We find the sufficient conditions which determine the weak recursion operator (for both types).

A recursion operator in a strong sense is the operator which transforms both gradients into gradients and symplectic forms into symplectic forms. The strong recursion operator generates simultaneously the infinite family of equations, starting from a given equation, and the infinite family of Hamiltonian structures for each equation from this family. The sufficient conditions for that the operator  $L$  be a strong recursion operator are given. The so-called Nijenhuis equation for  $L$  plays an important role in the theory of weak and strong recursion operators. The Hamiltonian systems both with finite and continual numbers of the degrees of freedom are considered. The

operator (1.2) is an example of the strong recursion operator.

In our paper a method is also proposed for the calculation of recursion operators in the theory of nonlinear waves in space of arbitrary dimension. This method is based on the subsequent use of the expansion of all quantities into a power series over the fields and of the momentum representation instead of a coordinate one. As a result, the equations in variational derivatives, which determine the recursion operator, convert into the system of algebraic functional equations for the coefficients of the expansions of the recursion operator and Hamiltonian. Some equations from this system offers the possibility of calculating the recursion operator via the Hamiltonian of a nonlinear equation.

In the paper we show that any Hamiltonian system of nonlinear waves possesses a formal recursion operator. In the general case, such a recursion operator is the singular operator. In the one-dimensional space there exist Hamiltonian systems which possess the regular recursion operator, i.e. the operator which generates the family of regular Hamiltonians. For some nonlinear equations the recursion operator can be a finite-order polynomial on the fields. The Hamiltonian of such an equation should satisfy a certain system of equations. In particular, it is shown that the only one-dimensional equation with three-linear Hamiltonian, which possesses a recursion operator, linear on field, is the KdV equation.

It is shown that the nonlinear evolution equations in two- and higher dimensional spaces have no regular recursion operators. In particular, the Kadomtsev-Petviashvili equation has no the regular recursion operator. Thus, the regular recursion operator is a purely one-dimensional phenomenon.

All these properties of recursion operators are closely connected with the formal canonical equivalence of nonlinear equations with the Hamiltonians, which are the "entire" functionals on fields, to linear equations. Performing the inverse canonical transformation from the linear equation to the initial nonlinear equation, we simultaneously obtain the Hamiltonian of the equation and the expression for the recursion operator. The regularity problem of the recursion operator is associated now with

the regularity problem of the linearizing canonical transformation.

The paper is written with the use of elementary methods only. The presented results can be, however, formulated in the invariant form as well.

The paper is organized as follows. The notions of H-weak and  $\Omega$ -weak recursion operators are introduced in the second section. In the third section the sufficient conditions for that the operator  $L_H$  be a H-weak recursion operator are found. In section 4 the necessary and sufficient conditions for that the operator  $L_\Omega$  be a  $\Omega$ -weak recursion operator are formulated. The strong recursion operator and the conditions which determine this operator are considered in section 5. Recursion operators in the theory of nonlinear waves (i.e. for systems with continual number of the degrees of freedom) and the conditions which define these operators are discussed in section 6. Section 7 is devoted to the use of the expansion into a power series over the fields and of the momentum representation for the calculation of recursion operators. The problem of existence of regular recursion operators in one- and multidimensional spaces is discussed in section 8.

## II. Weak recursion operators

In this and next sections we will consider Hamiltonian systems for the finite degrees of freedom. We would like to remind that a system of differential equations, which are defined on the  $2N$ -dimensional phase space, is called the Hamiltonian system if it may be represented in certain local coordinates as follows: (see e.g. [31])

$$\Omega \dot{x} = \nabla H \quad (2.1)$$

where  $x = (x^1, \dots, x^{2N})$ ,  $\nabla = (\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{2N}})$ ,  $\dot{x} = \frac{dx}{dt}$ ,  $H$  is a function and  $\Omega$  is a nondegenerate skewsymmetric ( $\Omega_{ik} = -\Omega_{ki}$ ) matrix which obeys the closeness condition

$$\frac{\partial \Omega_{ij}}{\partial x^k} + \frac{\partial \Omega_{ki}}{\partial x^j} + \frac{\partial \Omega_{jk}}{\partial x^i} = 0. \quad (2.2)$$

By virtue of the nondegenerateness of  $\Omega$ , equation (2.1) can be also represented as

$$\dot{x} = \Omega^{-1} \nabla H = \{x, H\} \quad (2.3)$$

where  $\{, \}$  denotes a Poisson bracket

$$\{F, H\} = \frac{\partial F}{\partial x^i} (\Omega^{-1})^{ik} \frac{\partial H}{\partial x^k}. \quad (2.4)$$

Here and below the summation is performed over repeated indices. It is well known also that locally by appropriate change of coordinates one can convert the form  $\Omega$  into the canonical one

$$\Omega_{(0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ where } 1 \text{ is an identical } N \times N \text{ matrix (Darboux theorem).}$$

So let we have a certain equation (2.1) with concrete Hamiltonian  $H$  and symplectic form  $\Omega$ . What is the way in which one can recurrently multiply this equation without leaving the class of Hamiltonian equations? It is easy to see that one can do this in two ways. The first way is to multiply the right-hand side of (2.1), i.e. the gradient  $\nabla H$ . The second way is to multiply the left-hand side, i.e. the symplectic form  $\Omega$ .

Definition. We will refer to the operator  $L_H$  as the H-weak recursion operator if any its power converts the gradient of  $H$  (for  $H \neq \text{const}$ ) into the gradients:  $L_H^n \nabla H = \nabla H_n$ ,  $n = 1, 2, 3, \dots$ . The operator  $L_\Omega$  is referred to as the  $\Omega$ -weak recursion operator if any its power converts some symplectic form  $\Omega$  into the symplectic forms:  $L_\Omega^n \Omega = \Omega_n$ ,  $n = 1, 2, 3, \dots$ .

Making use of the weak recursion operators, one can construct the following infinite families of equations, starting from equation (2.1):

$$\Omega \dot{x} = \nabla H_n = L_H^n \nabla H \quad n = 0, 1, 2, \dots \quad (2.5)$$

and

$$L_\Omega^n \Omega \dot{x} = \Omega_n \dot{x} = \nabla H \quad n = 0, 1, 2, \dots \quad (2.6)$$

The equations (2.5) are Hamiltonian ones with respect to the same symplectic form  $\Omega$  and different Hamiltonians  $H_n$ . The equations (2.6) are Hamiltonian ones with respect to the same Hamiltonian  $H$  and different symplectic forms  $\Omega_n$ . It is clear that any entire function of the recursion operator is the recursion operator of the same type, too.

Combining these two ways, one can obtain the most general family of the equations

$$\varphi(L_\Omega) \Omega \dot{x} = \Omega_\varphi \dot{x} = \nabla H_f = f(L_H) \nabla H \quad (2.7)$$

which are associated with equation (2.1). Here  $\varphi(L_\Omega)$  and  $f(L_H)$  are arbitrary entire functions. For  $\det L_\Omega \neq 0$ , equations (2.7) can be also represented in the form

$$\dot{x} = \Omega_\varphi^{-1} \nabla H_f = \Omega^{-1} \varphi^{-1}(L_\Omega) f(L_H) \nabla H. \quad (2.8)$$

### III. H-weak recursion operator

Here we find the conditions which determine weak recursion operators. Firstly, we consider the H-weak recursion operator. Let us note, first of all, that in order that the operator  $L_H$  transform a given gradient into gradients ( $L_H^n \nabla H = \nabla H_n$ ) it is necessary and sufficient that it satisfy the equation

$$\frac{\partial}{\partial x^j} \left( (L_H^n)_i^k \frac{\partial H}{\partial x^k} \right) = \frac{\partial}{\partial x^i} \left( (L_H^n)_j^k \frac{\partial H}{\partial x^k} \right). \quad (3.1)$$

The necessity of the condition (3.1) is obvious: it is the equality of cross derivatives which follows from  $(L_H^n)_i \frac{\partial H}{\partial x^k} = \frac{\partial H_n}{\partial x^i}$ . The sufficientness follows from the well known statement that for any simply connected manifold the equation  $\frac{\partial a_i}{\partial x^k} - \frac{\partial a_k}{\partial x^i} = 0$  implies  $a_i = \frac{\partial \varphi}{\partial x^i}$  (see e.g. [31]).

Proposition 3.1. The operator  $L_H$  is a H-weak recursion operator if it satisfies the system of equations

$$\frac{\partial L_{Hn}^m}{\partial x^k} \frac{\partial H}{\partial x^m} + L_{Hn}^m \frac{\partial^2 H}{\partial x^k \partial x^m} - \frac{\partial L_{Hk}^m}{\partial x^n} \frac{\partial H}{\partial x^m} - L_{Hk}^m \frac{\partial^2 H}{\partial x^n \partial x^m} = 0, \quad (3.2)$$

$$\left( \frac{\partial L_{Hi}^n}{\partial x^k} - \frac{\partial L_{Hk}^n}{\partial x^i} \right) L_{Hn}^m + L_{Hi}^n \frac{\partial L_{Hk}^m}{\partial x^n} - L_{Hk}^n \frac{\partial L_{Hi}^m}{\partial x^n} = 0. \quad (3.3)$$

Proof. The condition (3.2) means  $L_H \nabla H = \nabla H_x$ . Let us prove that by virtue of (3.2) and (3.3) we also have  $L_H^2 \nabla H = \nabla H_2$ . Let us multiply equation (3.2) by  $L_{Hi}^n$  and sum over  $n$ . We obtain

$$\begin{aligned} L_{Hi}^n \frac{\partial L_{Hn}^m}{\partial x^k} \frac{\partial H}{\partial x^m} + L_{Hi}^n L_{Hn}^m \frac{\partial^2 H}{\partial x^k \partial x^m} &= \\ = L_{Hi}^n \frac{\partial L_{Hk}^m}{\partial x^n} \frac{\partial H}{\partial x^m} + L_{Hi}^n L_{Hk}^m \frac{\partial^2 H}{\partial x^n \partial x^m}. \end{aligned} \quad (3.4)$$

Then multiplying (3.3) by  $\frac{\partial H}{\partial x^m}$  and summing over  $m$ , we find

$$\begin{aligned} \frac{\partial L_{Hi}^n}{\partial x^k} L_{Hn}^m \frac{\partial H}{\partial x^m} + L_{Hi}^n \frac{\partial L_{Hk}^m}{\partial x^n} \frac{\partial H}{\partial x^m} &= \\ = \frac{\partial L_{Hk}^n}{\partial x^i} L_{Hn}^m \frac{\partial H}{\partial x^m} + L_{Hk}^n \frac{\partial L_{Hi}^m}{\partial x^n} \frac{\partial H}{\partial x^m}. \end{aligned} \quad (3.5)$$

Further let us sum equation (3.5) with equation (3.4) and subtract equation (3.4) with the substitution  $i \leftrightarrow k$  from

the derived equation. As a result, we have

$$\begin{aligned} \frac{\partial L_{Hi}^n}{\partial x^k} L_{Hn}^m \frac{\partial H}{\partial x^m} + L_{Hi}^n \frac{\partial L_{Hn}^m}{\partial x^k} \frac{\partial H}{\partial x^m} + L_{Hi}^n L_{Hn}^m \frac{\partial^2 H}{\partial x^k \partial x^m} &= \\ = \frac{\partial L_{Hk}^n}{\partial x^i} L_{Hn}^m \frac{\partial H}{\partial x^m} + L_{Hk}^n \frac{\partial L_{Hn}^m}{\partial x^i} \frac{\partial H}{\partial x^m} + L_{Hk}^n L_{Hn}^m \frac{\partial^2 H}{\partial x^i \partial x^m}, \end{aligned} \quad (3.6)$$

i.e.

$$\frac{\partial}{\partial x^k} \left( (L_H^2)_i^m \frac{\partial H}{\partial x^m} \right) = \frac{\partial}{\partial x^i} \left( (L_H^2)_k^m \frac{\partial H}{\partial x^m} \right). \quad (3.7)$$

From (3.7) it follows that

$$(L_H^2)_i^m \frac{\partial H}{\partial x^m} = \frac{\partial H_2}{\partial x^i},$$

i.e.  $L_H^2 \nabla H = \nabla H_2$ , where  $H_2$  is a certain function.

Further let us show that  $L_H^3 \nabla H = \nabla H_3$  too. Multiplying (3.3) by  $L_{Hm}^e \frac{\partial H}{\partial x^e}$  and summing over  $m$ , we obtain

$$\begin{aligned} \frac{\partial L_{Hi}^n}{\partial x^k} L_{Hn}^m L_{Hm}^e \frac{\partial H}{\partial x^e} - \frac{\partial L_{Hk}^n}{\partial x^i} L_{Hn}^m L_{Hm}^e \frac{\partial H}{\partial x^e} + \\ + L_{Hi}^n \frac{\partial L_{Hk}^m}{\partial x^n} L_{Hm}^e \frac{\partial H}{\partial x^e} - L_{Hk}^n \frac{\partial L_{Hi}^m}{\partial x^n} L_{Hm}^e \frac{\partial H}{\partial x^e} &= 0. \end{aligned} \quad (3.8)$$

Then we multiply equation (3.6) (with the substitution  $i \rightarrow \rho$ ) by  $L_{Hi}^e$  and sum over  $\rho$ . Summing the obtained equation with equation (3.8) and using (3.2), we find

$$\frac{\partial L_{Hi}^n}{\partial x^k} (L_H^2)_n^e \frac{\partial H}{\partial x^e} + L_{Hi}^n \frac{\partial L_{Hn}^m}{\partial x^k} L_{Hm}^e \frac{\partial H}{\partial x^e} + (L_H^2)_i^m \frac{\partial L_{Hm}^e}{\partial x^k} \frac{\partial H}{\partial x^e} +$$

$$+ (L_H^3)_i^n \frac{\partial^2 H}{\partial x^k \partial x^n} - \frac{\partial L_{Hk}^n}{\partial x^i} (L_H^2)_n^e \frac{\partial H}{\partial x^e} - L_{Hk}^n \frac{\partial L_{Hn}^m}{\partial x^i} L_{Hm}^e \frac{\partial H}{\partial x^e} -$$

$$- (L_H^2)_k^m \frac{\partial L_{Hm}^e}{\partial x^i} \frac{\partial H}{\partial x^e} - (L_H^3)_k^n \frac{\partial^2 H}{\partial x^i \partial x^n} = 0,$$

i.e.

$$\frac{\partial}{\partial x^k} \left( (L_H^3)_i^n \frac{\partial H}{\partial x^n} \right) = \frac{\partial}{\partial x^i} \left( (L_H^3)_k^n \frac{\partial H}{\partial x^n} \right). \quad (3.9)$$

Thus  $L_H^3 \nabla H = \nabla H_3$  where  $H_3$  is a certain function.

In a similar manner, one can prove by induction that the equality (3.1) is valid for  $n = 4, 5, 6, \dots$  too, i.e.  $L_H^n \nabla H = \nabla H_n$ ,  $n = 1, 2, 3, 4, \dots$ . The proposition is proved.

Emphasize the important role of the quadratic equation (3.3). This equation together with the condition  $L_H \nabla H = \nabla H_2$  is equivalent to the equalities  $L_H^n \nabla H = \nabla H_n$  for any  $n = 1, 2, 3, \dots$ . Note, however, that equation (3.3) does not follow from any finite subsystem of equations  $L_H^n \nabla H = \nabla H_n$  ( $n = 1, \dots, M$ ).

Let us now discuss some properties of equations (2.5). In the general case the flows which are generated by Hamiltonians  $H_n$  do not commute with the initial flow (2.1) since ( $H_0 = H$ )

$$\{H_n, H_0\} = \frac{\partial H_n}{\partial x^i} (\Omega^{-1})^{ik} \frac{\partial H_0}{\partial x^k} = \frac{\partial H}{\partial x^m} (L_H^n)_i^m (\Omega^{-1})^{ik} \frac{\partial H}{\partial x^k} \neq 0. \quad (3.10)$$

Let the operator  $L_H$  satisfy the additional constraint  $(L_H \Omega)^T = -L_H \Omega$  where  $A^T$  denotes a transposed matrix  $A$ . Using this constraint, we have

$$\frac{\partial H}{\partial x^m} (L_H^n)_i^m (\Omega^{-1})^{ik} \frac{\partial H}{\partial x^k} = \frac{\partial H}{\partial x^m} (\Omega^{-1})^{mi} (L_H^n)_i^k \frac{\partial H}{\partial x^k} =$$

$$= - \frac{\partial H}{\partial x^k} (L_H^n)_i^k (\Omega^{-1})^{im} \frac{\partial H}{\partial x^m},$$

i.e.  $\frac{\partial H}{\partial x^m} (L_H^n)_i^m (\Omega^{-1})^{ik} \frac{\partial H}{\partial x^k} = 0$ . Therefore  $\{H_n, H_0\} = 0$ ,  $n = 1, 2, 3, \dots$ . Analogously, one can show that  $\{H_n, H_m\} = 0$  ( $n, m = 0, 1, 2, \dots$ ), i.e. in the case  $(L_H \Omega)_{ik} = -(L_H \Omega)_{ki}$  all the flows from the family (2.5) commute to each other.

Thus, in the case  $(L_H \Omega)^T = -L_H \Omega$  the family of Hamiltonians  $H_n$  is the infinite family of the integrals of motion for any equation of the form (2.5). Each integral of motion  $H_n$  is connected with the one-parameter symmetry group of equations (2.5) and, in particular, of the initial equation (2.1). In the infinitesimal form these symmetry transformations are  $(x' = x + \delta x)$

$$\delta_n x = \varepsilon_n \Omega^{-1} \nabla H_n = \varepsilon_n \Omega^{-1} L_H^n \nabla H \quad (3.11)$$

$$n = 0, 1, 2, 3, \dots$$

where  $\varepsilon_n$  is the transformation parameter. If  $(L_H \Omega)^T = -L_H \Omega$  equations (2.5) and the symmetry transformations (3.11) can be also represented in the form

$$\dot{x} = (L_H^T)^n \Omega^{-1} \nabla H$$

and

$$\delta_n x = \varepsilon_n (L_H^T)^n \Omega^{-1} \nabla H$$

$$n = 0, 1, 2, 3, \dots$$

#### IV. $\Omega$ -weak recursion operator

According to the definition, the operator  $L_\Omega$  is a  $\Omega$ -weak recursion operator if



$$\frac{\partial(L_{\Omega}^n \Omega)_{ij}}{\partial x^k} + \frac{\partial(L_{\Omega}^n \Omega)_{ki}}{\partial x^j} + \frac{\partial(L_{\Omega}^n \Omega)_{jk}}{\partial x^i} = 0 \quad (4.1)$$

and

$$(L_{\Omega}^n \Omega)^T = -L_{\Omega}^n \Omega \quad (4.2)$$

for all  $n = 0, 1, 2, \dots$

One can show by straightforward calculations that if equations (4.1) and (4.2) are satisfied for  $n = 0, 1, 2$  then they hold for  $n = 3$ , too. As a result, they are satisfied for any  $n$ . So we have

Proposition 4.1. If together with the form  $\Omega$  the forms  $L_{\Omega} \Omega$  and  $L_{\Omega}^2 \Omega$  are closed and skewsymmetric, then the operator  $L_{\Omega}$  is a  $\Omega$ -weak recursion operator.

Note that all the conditions (4.2) are satisfied if  $\Omega^T = -\Omega$  and  $(L_{\Omega} \Omega)^T = -L_{\Omega} \Omega$ .

If additionally  $\det L_{\Omega} \neq 0$  then the closeness and skewsymmetry of the forms  $\Omega$ ,  $L_{\Omega} \Omega$ ,  $L_{\Omega}^2 \Omega$  lead also to the closeness and skewsymmetry of the forms  $L_{\Omega}^{-n} \Omega$ ,  $n = 1, 2, 3, \dots$ . Indeed, choosing the form  $\Omega_2 = L_{\Omega}^2 \Omega$  as an initial one and using the closeness and skewsymmetry of the forms  $L_{\Omega}^{-1} \Omega_2$ , and  $L_{\Omega}^{-2} \Omega_2$ , one can prove that the form  $L_{\Omega}^{-3} \Omega_2 = L_{\Omega}^{-1} \Omega$  is closed and skewsymmetric too. Further one can easily show that all the forms  $L_{\Omega}^{-n} \Omega_2 = L_{\Omega}^{2-n} \Omega$  ( $n = 3, 4, \dots$ ) are closed and skewsymmetric.

In the case  $\det L_{\Omega} \neq 0$  the closeness and skewsymmetry of one of the following two sets of the forms:  $L_{\Omega}^{-1} \Omega, \Omega$ ,  $L_{\Omega} \Omega$  or  $L_{\Omega}^{-2} \Omega, L_{\Omega}^{-1} \Omega, \Omega$  are the sufficient conditions in order that the operator  $L_{\Omega}$  be a  $\Omega$ -weak recursion operator too.

In Proposition 4.1 and subsequent results, one can start from any given symplectic form  $\Omega$ . To simplify the calculations it is convenient, however, to choose the local coordinates in such a way that  $\Omega = \Omega_{(0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . This does not reduce

the generality of the results obtained since equations (4.1) and (4.2) are of invariant character.

In the case  $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  equations (4.1) and (4.2) for  $n = 0, 1, 2$  are

$$\frac{\partial L_{\Omega i}^n}{\partial x^l} \Omega_{nm} + \frac{\partial L_{\Omega e}^n}{\partial x^m} \Omega_{ni} + \frac{\partial L_{\Omega m}^n}{\partial x^i} \Omega_{ne} = 0, \quad (4.3)$$

$$\frac{\partial (L_{\Omega}^2)_i^n}{\partial x^l} \Omega_{nm} + \frac{\partial (L_{\Omega}^2)_e^n}{\partial x^m} \Omega_{ni} + \frac{\partial (L_{\Omega}^2)_m^n}{\partial x^i} \Omega_{ne} = 0 \quad (4.4)$$

and

$$\Omega_{nk} L_{\Omega e}^k = L_{\Omega n}^k \Omega_{ke}. \quad (4.5)$$

Multiplying (4.3) by  $L_{\Omega k}^m$ , summing over  $m$  and taking into account (4.5), one obtains

$$\frac{\partial L_{\Omega i}^n}{\partial x^l} L_{\Omega n}^m \Omega_{mk} + L_{\Omega k}^n \frac{\partial L_{\Omega n}^m}{\partial x^i} \Omega_{me} = -L_{\Omega k}^n \frac{\partial L_{\Omega e}^m}{\partial x^i} \Omega_{mi}. \quad (4.6)$$

Using (4.6), it is easy to see that equation (4.4) is equivalent to the following one:

$$L_{\Omega i}^n \frac{\partial L_{\Omega k}^m}{\partial x^n} \Omega_{me} + L_{\Omega e}^n \frac{\partial L_{\Omega i}^m}{\partial x^n} \Omega_{mk} + L_{\Omega k}^n \frac{\partial L_{\Omega e}^m}{\partial x^n} \Omega_{mi} = 0. \quad (4.7)$$

Let us transform the second and third terms in (4.7), using the equality (4.5) and  $\Omega_{nt} (\Omega^{-1})^{tp} = \delta_n^p$ . As a result, we have

$$L_{\Omega i}^n \frac{\partial L_{\Omega k}^m}{\partial x^n} \Omega_{me} - \frac{\partial L_{\Omega i}^p}{\partial x^n} \Omega_{kp} (\Omega^{-1})^{nt} L_{\Omega t}^m \Omega_{me} - L_{\Omega k}^n \frac{\partial L_{\Omega e}^m}{\partial x^n} \Omega_{me} = 0. \quad (4.8)$$

Hence

$$L_{\Omega i}^n \frac{\partial L_{\Omega k}^m}{\partial x^n} - \frac{\partial L_{\Omega i}^p}{\partial x^n} \Omega_{kp} (\Omega^{-1})^{nt} L_{\Omega t}^m - L_{\Omega k}^n \frac{\partial L_{\Omega i}^m}{\partial x^n} = 0. \quad (4.9)$$

We note now that in virtue of (4.5) equation (4.3) is equivalent to

$$\frac{\partial L_{\Omega i}^n}{\partial x^e} \Omega_{nm} + \frac{\partial L_{\Omega e}^n}{\partial x^m} \Omega_{ni} - \frac{\partial L_{\Omega e}^n}{\partial x^i} \Omega_{nm} = 0. \quad (4.10)$$

If one multiplies (4.10) by  $(\Omega^{-1})^{mk}$  and sums over  $m$ , one obtains

$$\frac{\partial L_{\Omega i}^k}{\partial x^e} - \frac{\partial L_{\Omega e}^k}{\partial x^i} = - \frac{\partial L_{\Omega e}^n}{\partial x^m} \Omega_{ni} (\Omega^{-1})^{mk} = - \frac{\partial L_{\Omega i}^n}{\partial x^m} \Omega_{en} (\Omega^{-1})^{mk}. \quad (4.11)$$

Substitution of (4.11) into (4.9), eventually, gives

$$L_{\Omega i}^n \frac{\partial L_{\Omega k}^m}{\partial x^n} + \left( \frac{\partial L_{\Omega i}^t}{\partial x^k} - \frac{\partial L_{\Omega k}^t}{\partial x^i} \right) L_{\Omega t}^m - L_{\Omega k}^n \frac{\partial L_{\Omega i}^m}{\partial x^n} = 0. \quad (4.12)$$

So the equation, which contains only the recursion operator  $L_{\Omega}$ , follows from equations (4.3)-(4.5).

**Proposition 4.2.** If operator  $L_{\Omega}$  satisfies the system of equations

$$\left( \frac{\partial L_{\Omega i}^m}{\partial x^k} - \frac{\partial L_{\Omega k}^m}{\partial x^i} \right) L_{\Omega m}^e + L_{\Omega i}^n \frac{\partial L_{\Omega k}^e}{\partial x^n} - L_{\Omega k}^n \frac{\partial L_{\Omega i}^e}{\partial x^n} = 0, \quad (4.13)$$

$$\frac{\partial (L_{\Omega \Omega})_{ik}}{\partial x^e} + \frac{\partial (L_{\Omega \Omega})_{ei}}{\partial x^k} + \frac{\partial (L_{\Omega \Omega})_{ke}}{\partial x^i} = 0, \quad (4.14)$$

$$(L_{\Omega \Omega})_{ke} = -(L_{\Omega \Omega})_{ek} \quad (4.15)$$

then it is a  $\Omega$ -weak recursion operator.

Proof. Let  $\Omega = \Omega_{(0)}$ . The conditions (4.13)-(4.15) are equivalent to the conditions (4.3)-(4.6). Indeed, from (4.12) and (4.11) one gets (4.9). Multiplying (4.9) by  $\Omega_{me}$ , we obtain (4.8). Equation (4.7) follows from (4.8) and is equivalent to (4.4) due to (4.6). The conditions (4.13)-(4.15) are, therefore, equivalent to the conditions of Proposition 4.1.

Let us consider the conditions (4.13) and (4.14) in more detail for the case  $\det L_{\Omega} \neq 0$ . At first sight, these conditions are not necessary. Indeed, if one takes instead of  $\Omega, L_{\Omega} \Omega, L_{\Omega}^2 \Omega$  the other sets of three closed forms:  $L_{\Omega}^{-2} \Omega, L_{\Omega}^{-1} \Omega, \Omega$  or  $L_{\Omega}^{-1} \Omega, \Omega, L_{\Omega} \Omega$  then instead of (4.13) and (4.14) we will have the analogous conditions with the substitution  $L_{\Omega} \rightarrow L_{\Omega}^{-1}$ .

However, it is worth noting the following. Firstly, multiplying equation (4.13) for  $L_{\Omega}^{-1}$  by  $L_{\Omega}^i L_{\Omega}^k L_{\Omega}^e$ , summing over  $i, k, e$  and taking into account (4.13), one gets equation (4.13) for  $L_{\Omega}$ . So equation (4.13) for  $L_{\Omega}^{-1}$  is equivalent to the same equation for  $L_{\Omega}$ . In other words, if  $L_{\Omega}$  is the solution of (4.13), then  $L_{\Omega}^{-1}$  is the solution, too.

Secondly, the conditions of closeness and skewsymmetry of the form  $L_{\Omega}^{-1} \Omega$  together with equation (4.13) are equivalent to the conditions of closeness and skewsymmetry of the form  $L_{\Omega} \Omega$ . Indeed, putting  $\Omega = \Omega_{(0)}$ , multiplying (4.14) for  $L_{\Omega}^{-1} \Omega$  by  $L_{\Omega}^i L_{\Omega}^k L_{\Omega}^e$ , summing over  $i, k, e$  and using (4.15), we obtain equation (4.7). Equation (4.9) follows from (4.7). Using (4.12) one gets (4.11). This equation is equivalent to (4.10) and, hence, to equation (4.3).

Thus, at  $\det L_{\Omega} \neq 0$  the conditions (4.13)-(4.15) for  $L_{\Omega}^{-1}$  are equivalent to the conditions (4.13)-(4.15) for  $L_{\Omega}$ .

So in the case  $\det L_{\Omega} \neq 0$  we have

Theorem 4.1. The conditions (4.13)-(4.15) are necessary and sufficient conditions in order that the operator  $L_\Omega$  be a  $\Omega$ -weak recursion operator.

The invariant form of equation (3.9) (or (4.13)) is the following:

$$[L^T \xi, L^T \eta] - L^T [L^T \xi, \eta] - L^T [\xi, L^T \eta] + L^{T^2} [\xi, \eta] = 0 \quad (4.16)$$

where  $L^T$  is the transposed matrix  $L$ ,  $\xi$  and  $\eta$  are arbitrary vector fields, and  $[\xi, \eta]$  denotes the standard commutator of vector fields (see e.g. [31]):  $[\xi, \eta]_i = \sum_k \left( \frac{\partial \xi_i}{\partial x^k} \eta^k - \frac{\partial \eta_i}{\partial x^k} \xi^k \right)$ . Following the paper [26] where equation (4.16) was considered for the first time, we will refer to equation (4.16) as the Nijenhuis equation. Note that for  $\Omega$ -weak recursion operator one has  $L_\Omega^T = \Omega^{-1} L_\Omega \Omega$  due to (4.2).

An equation of the form (4.16) has been also considered in [27-30].

#### V. Strong recursion operator

Let us consider now the situation when the operator  $L$  is a recursion one both in  $H$  and  $\Omega$  senses.

Definition. Operator  $L$  is the strong recursion operator if any its power transforms gradient into gradients ( $L^n \nabla H = \nabla H_n$ ) and symplectic form into symplectic forms ( $L^n \Omega = \Omega_n$ ).

Possessing simultaneously the properties of both weak recursion operators, the strong recursion operator generates both the infinite family of Hamiltonians  $H_n$  and the infinite family of symplectic forms  $\Omega_n = L^n \Omega$ . Equations (2.5), (2.6) or (2.8) which are generated by a strong recursion operator have all previous properties and some new ones.

Let us have some equation  $\dot{x} = \Omega^{-1} \nabla H$  from the family of equations (2.8), and let  $\det L \neq 0$ . By virtue of the properties of the strong recursion operator, we have

$$\dot{x} = \Omega^{-1} \nabla H = (L^n \Omega)^{-1} L^n \nabla H = \Omega_n^{-1} \nabla H_n \quad (5.1)$$

where  $n$  is any integer and  $\Omega_n$  are the closed symplectic forms.

So any equation generated by the strong recursion operator is a Hamiltonian one with respect to the infinite set of Hamiltonian structures (pairs  $\Omega_n, H_n$ ).

We denote the Poisson bracket which corresponds to the form  $\Omega_n$  as  $\{, \}_n : \{F, H\}_n = \frac{\partial F}{\partial x^i} (\Omega_n^{-1})^{ik} \frac{\partial H}{\partial x^k}$ . Let us calculate  $\{H_{n_1}, H_{n_2}\}_{n_3}$ . Taking into account that  $(L\Omega)^T = -L\Omega$ , we have

$$\begin{aligned} \{H_{n_1}, H_{n_2}\}_{n_3} &= (L^{n_1})_i^m \frac{\partial H}{\partial x^m} (\Omega^{-1})^{ie} (L^{n_2-n_3})_e^k \frac{\partial H}{\partial x^k} = \\ &= \frac{\partial H}{\partial x^m} (\Omega^{-1})^{mi} (L^{n_1+n_2-n_3})_i^k \frac{\partial H}{\partial x^k}. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial H}{\partial x^m} (\Omega^{-1})^{mi} (L^n)_i^k \frac{\partial H}{\partial x^k} &= \frac{\partial H}{\partial x^m} (L^n)_i^m (\Omega^{-1})^{ik} \frac{\partial H}{\partial x^k} = \\ &= - \frac{\partial H}{\partial x^k} (\Omega^{-1})^{ki} (L^n)_i^m \frac{\partial H}{\partial x^m}. \end{aligned}$$

Therefore,  $\frac{\partial H}{\partial x^m} (\Omega^{-1})^{mi} (L^n)_i^k \frac{\partial H}{\partial x^k} = 0$  and, as a consequence,  $\{H_{n_1}, H_{n_2}\}_{n_3} = 0$  for any  $n_1, n_2, n_3$ .

Thus, all Hamiltonians  $H_n$  which are generated by the strong recursion operator are in involution with respect to any Poisson bracket  $\{, \}_n$  generated by this operator.

It is clear also that if the initial equation (2.1) admits the strong recursion operator  $L$ , then any equation of the form (2.8) possesses the same strong recursion operator.

So the Hamiltonian equations, which admit strong recursion operator, have a very special structure. Firstly, they possess the infinite set of the integrals of motion in involution and they are Hamiltonian ones with respect to the infinite family of Hamiltonian structures. Secondly, the infinite families of equations are associated with such equations: any equation of these families has the same properties as the initial one.

In other words, the strong recursion operator generates the infinite family of Hamiltonian structures  $(\Omega_n, H_m; n, m = 0, \pm 1, \pm 2, \dots)$  from the initial Hamiltonian system. Each of these Hamiltonian structures determines the dynamical system (flow). The Hamiltonian structures  $(\Omega_n, H_m)$  with the same value of  $n-m$  correspond to the same dynamical system. The family of Hamiltonians  $H_m$  forms the infinite set of common integrals of motion which are in involution with respect to any symplectic structure  $\Omega_n$ .

The examples of strong recursion operators are well known. In the case of a continual number of the degrees of freedom they are, for example, the operator (1.2) and equations (1.1), and the recursion operators which were calculated in [10-22]. Some properties of the equations which admit the strong recursion operator have been discussed as well.

The operator  $\tilde{L} = \Omega^{-1} L \Omega$  is closely connected with the recursion operator  $L$ . The operator  $\tilde{L}$  transforms vector fields into vector fields:  $\tilde{L}^n \Omega^{-1} \nabla H = \Omega^{-1} \nabla H_n$ . The equations (2.5)-(2.6) can be also represented in the form  $\dot{x}^i = \tilde{L}^n \Omega^{-1} \nabla H$ . Both operators  $L$  and  $\tilde{L}$  naturally appear in the approach which is based on the spectral problems (see e.g. [20, 21]).

Let us consider now the conditions which define a strong recursion operator. These conditions are obviously the join of the conditions which determine the  $H$ - and  $\Omega$ -weak recursion operators. Propositions (3.2) and (4.2) give rise to

Theorem 5.1. If operator  $L$  satisfies the system of equations

$$\left(\frac{\partial L_i^m}{\partial x^k} - \frac{\partial L_k^m}{\partial x^i}\right) L_m^e + L_i^n \frac{\partial L_k^e}{\partial x^n} - L_k^n \frac{\partial L_i^e}{\partial x^n} = 0, \quad (5.2)$$

$$\frac{\partial L_i^n}{\partial x^k} \frac{\partial H}{\partial x^n} + L_i^n \frac{\partial^2 H}{\partial x^k \partial x^n} - \frac{\partial L_k^n}{\partial x^i} \frac{\partial H}{\partial x^n} - L_k^n \frac{\partial^2 H}{\partial x^i \partial x^n} = 0, \quad (5.3)$$

$$\frac{\partial (L\Omega)_{ik}}{\partial x^e} + \frac{\partial (L\Omega)_{ei}}{\partial x^k} + \frac{\partial (L\Omega)_{ke}}{\partial x^i} = 0, \quad (5.4)$$

$$(L\Omega)_{ke} = -(L\Omega)_{ek} \quad (5.5)$$

then it is the strong recursion operator for the Hamiltonian system  $\Omega \dot{x} = \nabla H$ .

In the case  $\det L \neq 0$  the conditions (5.2), (5.4) and (5.5) are also necessary ones.

So the  $H$ -weak recursion operator becomes the strong recursion operator if it also transforms symplectic form  $\Omega$  into symplectic form. The  $\Omega$ -weak recursion operator becomes a strong one if it additionally converts the gradient  $\nabla H$  into the gradient  $(L\nabla H = \nabla H_s)$ .

We see that in the description of both the weak and strong recursion operators the quadratic equation (5.2) plays an important role. This equation is a very special one. It is a system of  $2N^2(2N-1)$  equations for  $(2N)^2$  quantities  $L_i^k$  ( $i, k = 1, \dots, 2N$ ). Nevertheless, this highly overdetermined (for  $N > 1$ ) system has a large class of solutions. As we have seen, if  $L$  is a solution of (5.2), then  $L^{-1}$  is a solution too. It is not difficult also to show that together with  $L$  the quantity  $1 - \lambda L$  is also the solution for any number  $\lambda$ . Hence,  $(1 - \lambda L)^{-1} = 1 + \lambda L + \lambda^2 L^2 + \dots$  is the solution of (5.2) too. The simplest solution of equation (5.2) is  $L_i^k = \delta_i^k a_i(x^i)$  where  $a_i(x^i)$  are arbitrary functions. In virtue of the invariance of equation (5.2) under the general coordinate transformations  $x^i \rightarrow x^{i'} = f^{i'}(x)$ , equation (5.2) has also the

solutions of the form  $L_i^k = \sum_{e=1}^{2N} a_e(x^e) \frac{\partial f^k(x)}{\partial x^e} \frac{\partial x^e}{\partial f^i(x)}$  where  $f^k(x)$  and  $a_e(x^e)$  are arbitrary functions.

In the conclusion of this section we compare the results of the present paper with those of the papers [26-28]. The key notion in the papers [26] was the notion of a Hamiltonian pair, i.e. two Hamiltonian operators such that any their linear superposition is a Hamiltonian operator too. In this approach the recursion operator appears as the "ratio" of two Hamiltonian operators from the Hamiltonian pair.

In our approach we deal with the recursion operator from the very beginning. If a system admits the  $\Omega$ -weak recursion operator  $L_\Omega$ , then all the forms  $L_\Omega^n \Omega$  ( $n = 0, 1, 2, \dots$ ) are closed. The form  $L_\Omega \Omega + \lambda L_\Omega^2 \Omega + \lambda^2 L_\Omega^3 \Omega + \dots = (1 - \lambda L_\Omega)^{-1} L_\Omega \Omega$ , where  $\lambda$  is any number, is closed too. Therefore the operator  $((1 - \lambda L_\Omega)^{-1} L_\Omega \Omega)^{-1} = (L_\Omega \Omega)^{-1} - \lambda \Omega^{-1}$  is a Hamiltonian one for any  $\lambda$ , i.e. the operators  $(L_\Omega \Omega)^{-1}$  and  $\Omega^{-1}$  form the Hamiltonian pair.

**Proposition 5.1** (Gelfand and Dorphman) In order that the Hamiltonian operators  $\Omega^{-1}$  and  $(L_\Omega \Omega)^{-1}$  form the Hamiltonian pair it is necessary and sufficient that the forms  $\Omega$ ,  $L_\Omega \Omega$ ,  $L^2 \Omega$  be the closed ones.

Sufficiency immediately follows from Proposition (4.1). Let us prove the necessity. The Jacoby identity for the Hamiltonian operator  $\Omega^{-1} + \lambda(L_\Omega \Omega)^{-1}$  leads to the closeness of the forms  $\Omega$ ,  $L_\Omega \Omega$  and also to the equation

$$(\Omega^{-1})^{im} \frac{\partial (L_\Omega \Omega)^{-1jk}}{\partial x^m} + (\Omega^{-1})^{km} \frac{\partial (L_\Omega \Omega)^{-1ij}}{\partial x^m} + (\Omega^{-1})^{jm} \frac{\partial (L_\Omega \Omega)^{-1ki}}{\partial x^m} = 0 \quad (5.6)$$

where we choose  $\Omega = \Omega_{(0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Multiplying (5.6) by  $(L_\Omega \Omega)_{il} (L_\Omega \Omega)_{pj} (L_\Omega \Omega)_{rk}$  and summing over  $l, j, k$  we obtain (4.7). By virtue of (4.3), the equality (4.7) is equivalent to (4.9), i.e. to the closeness condition for the

form  $L^2 \Omega$ . The l.h.s. of (5.6) is nothing but the Schouten bracket (see e.g. [26]) for the operators  $\Omega^{-1}$  and  $(L_\Omega \Omega)^{-1}$ . So the equality to zero of the Schouten bracket  $[\Omega^{-1}, (L_\Omega \Omega)^{-1}]$  is equivalent to the closeness of the forms  $L_\Omega \Omega$  and  $L^2 \Omega$ .

If the operator  $L$  is a  $\Omega$ -weak recursion operator, then any two operators  $(L^{n_1} \Omega)^{-1}$  and  $(L^{n_2} \Omega)^{-1}$  form the Hamiltonian pair. Indeed, in virtue of the closeness of the form

$$L^{n_1} \Omega + \lambda L^{2n_1 - n_2} \Omega + \lambda^2 L^{3n_1 - 2n_2} \Omega + \dots = (1 - \lambda L^{n_1 - n_2})^{-1} L^{n_1} \Omega$$

the operator  $((1 - \lambda L^{n_1 - n_2})^{-1} L^{n_1} \Omega)^{-1} = (L^{n_1} \Omega)^{-1} - \lambda (L^{n_2} \Omega)^{-1}$  is a Hamiltonian one for any  $\lambda$  and  $n_1, n_2$ .

Taking into account all these results, it is not difficult to see, that from the point of view of the construction of Hamiltonian systems the theorem 5.1. and theorem 3.4 from [26] are equivalent in essence.

The recursion operator, which is considered in [26] (regular operator), is the  $\Omega$ -weak recursion operator in our terminology.

The fact that the operator  $\Omega \tilde{L} \Omega^{-1}$ , where  $\tilde{L} \Omega^{-1} \nabla H = \Omega^{-1} \nabla H_x$ , transform gradients into gradients, i.e. it is the  $H$ -weak recursion operator, and also the fact that the eigenfunctions of this operator are gradients of its eigenvalues have been noted in [30].

## VI. Recursion operators in the theory of nonlinear waves

Generalization of the results of the previous sections to the theory of nonlinear waves, i.e. to the case of a continual number of the degrees of freedom, is obvious enough. Let we have the system of  $n$  fields  $u^1(x, t), \dots, u^n(x, t)$  in  $d$ -dimensional space  $(x = x_1, \dots, x_d)$ . Let us remind (see e.g.

[32,33]) that the system of  $N$  equations is a Hamiltonian one if it can be represented in the form

$$\int dx' \Omega_{\alpha\beta}(x, x') \frac{\partial u^\beta(x', t)}{\partial t} = \frac{\delta H}{\delta u^\alpha(x, t)} \quad (6.1)$$

$(\alpha = 1, \dots, n)$

where the Hamiltonian  $H$  is a certain functional on  $u^1, \dots, u^n$ ,  $\delta/\delta u$  denotes a variational derivative and  $\Omega_{\alpha\beta}(x, x')$  is a kernel of nondegenerate linear operator (which depends, in general, on  $u^1, \dots, u^n$ ) which satisfies the closeness

$$\frac{\delta \Omega_{\alpha\beta}(x, x')}{\delta u^\gamma(x'')} + \frac{\delta \Omega_{\gamma\alpha}(x'', x)}{\delta u^\beta(x')} + \frac{\delta \Omega_{\beta\gamma}(x', x'')}{\delta u^\alpha(x)} = 0 \quad (6.2)$$

and skewsymmetry conditions

$$\Omega_{\alpha\beta}(x, x') = -\Omega_{\beta\alpha}(x', x) \quad (6.3)$$

Similar to the case of a finite number of the degrees of freedom, we define the weak and strong recursion operators:

$$\int dx' (L_H^\alpha)^\beta(x, x') \frac{\delta H}{\delta u^\beta(x')} = \frac{\delta H_n}{\delta u^\alpha(x)}, \quad (6.4)$$

$$\int dx' (L_\Omega^\alpha)^\beta(x, x') \Omega_{\beta\gamma}(x', x'') = \Omega_{n\alpha\gamma}(x, x''). \quad (6.5)$$

The Propositions and Theorems of the previous sections take place for nonlinear waves, too. The only modification is concerned with the form of the corresponding equations. In particular, the analogs of equations (5.2)-(5.5) are

$$\int d\tilde{x} \left\{ \left( \frac{\delta L_\alpha^\delta(x, \tilde{x})}{\delta u^\beta(x')} - \frac{\delta L_\beta^\delta(x', \tilde{x})}{\delta u^\alpha(x)} \right) L_\delta^\gamma(\tilde{x}, x'') + \right.$$

$$\left. + L_\alpha^\delta(x, \tilde{x}) \frac{\delta L_\beta^\delta(x', x'')}{\delta u^\delta(\tilde{x})} - L_\beta^\delta(x', \tilde{x}) \frac{\delta L_\alpha^\delta(x, x'')}{\delta u^\delta(\tilde{x})} \right\} = 0, \quad (6.6)$$

$$\int d\tilde{x} \left\{ \frac{\delta L_\alpha^\rho(x, \tilde{x})}{\delta u^\beta(x')} \frac{\delta H}{\delta u^\rho(\tilde{x})} + L_\alpha^\rho(x, \tilde{x}) \frac{\delta^2 H}{\delta u^\beta(x') \delta u^\rho(\tilde{x})} - \right.$$

$$\left. - \frac{\delta L_\beta^\rho(x', \tilde{x})}{\delta u^\alpha(x)} \frac{\delta H}{\delta u^\rho(\tilde{x})} - L_\beta^\rho(x', \tilde{x}) \frac{\delta^2 H}{\delta u^\alpha(x) \delta u^\rho(\tilde{x})} \right\} = 0, \quad (6.7)$$

$$\frac{\delta (L\Omega)_{\alpha\beta}(x, x')}{\delta u^\gamma(x'')} + \frac{\delta (L\Omega)_{\gamma\alpha}(x'', x)}{\delta u^\beta(x')} + \frac{\delta (L\Omega)_{\beta\gamma}(x', x'')}{\delta u^\alpha(x)} = 0, \quad (6.8)$$

$$(L\Omega)_{\beta\alpha}(x', x) = -(L\Omega)_{\alpha\beta}(x, x') \quad (6.9)$$

where  $(L\Omega)_{\alpha\beta}(x, x') \stackrel{\text{def}}{=} \int dx'' L_\alpha^\gamma(x, x'') \Omega_{\gamma\beta}(x'', x')$

**Theorem 6.1.** If operator  $L$  satisfies equations (6.6), (6.7) then it is a  $H$ -weak recursion operator. If operator  $L$  satisfies equations (6.6), (6.8), (6.9), it is a  $\Omega$ -weak recursion operator. In the case when operator  $L$  satisfies the whole system of equations (6.6)-(6.9), it is the strong recursion operator.

The strong recursion operator  $L$  generates, starting from (6.1), the infinite family of nonlinear evolution equations

$$\frac{\partial u^\alpha(x, t)}{\partial t} = \int dx' (\Omega^{-1} \varphi(L))^{\alpha\beta}(x, x') \frac{\delta H}{\delta u^\beta(x', t)} \quad (6.10)$$

where  $\varphi(L)$  is any entire (meromorphic for  $\det L \neq 0$ ) scalar function. Each of equations (6.10) possesses the infinite set of the integrals of motion  $H_n$  and is a Hamiltonian one with respect to the infinite family of Poisson brackets of the form

$$\{F, H\}_\varphi = \int dx dx' \frac{\delta F}{\delta u^\alpha(x)} (\Omega^{-1} \varphi(L))^{\alpha\beta}(x, x') \frac{\delta H}{\delta u^\beta(x')} \quad (6.11)$$

where  $f(L)$  is an arbitrary entire (meromorphic for  $\det L \neq 0$ ) function.

Taking into account our further constructions, we rewrite the formulae given here in the momentum representation. Performing the Fourier transform

$$u^\alpha(x, t) = (2\pi)^{-\frac{d}{2}} \int dp a_p^\alpha(t) \exp(ipx),$$

$$\Omega_{\alpha\beta}(x, x') = (2\pi)^{-d} \int dp dp' \Omega_{\alpha\beta, pp'} \exp(ipx + ip'x'),$$

$$L_\alpha^\beta(x, x') = (2\pi)^{-d} \int dp dp' L_{\alpha, pp'}^\beta \exp(ipx + ip'x') \quad (6.12)$$

$$(\rho = (\rho_1, \dots, \rho_d), \rho x \stackrel{\text{def}}{=} \rho_1 x_1 + \dots + \rho_d x_d)$$

we get the closeness condition

$$\frac{\delta \Omega_{\alpha\beta, pq}}{\delta a_{-k}^\gamma} + \frac{\delta \Omega_{\gamma\alpha, kp}}{\delta a_{-q}^\beta} + \frac{\delta \Omega_{\beta\gamma, qk}}{\delta a_{-p}^\alpha} = 0 \quad (6.13)$$

and the skewsymmetry condition

$$\Omega_{\alpha\beta, pq} = -\Omega_{\beta\alpha, qp} \quad (6.14)$$

Equations (6.6)-(6.9) in the momentum representation are of the form

$$\int d\tilde{p} \left\{ L_{\alpha, p\tilde{p}}^\beta \frac{\delta L_{\beta, qk}^\gamma}{\delta a_{\tilde{p}}^\alpha} - L_{\beta, q\tilde{p}}^\alpha \frac{\delta L_{\alpha, pk}^\gamma}{\delta a_{\tilde{p}}^\beta} + \left( \frac{\delta L_{\alpha, p\tilde{p}}^\beta}{\delta a_{-q}^\beta} - \frac{\delta L_{\beta, q\tilde{p}}^\alpha}{\delta a_{-p}^\alpha} \right) L_{\beta, -\tilde{p}k}^\gamma \right\} = 0, \quad (6.15)$$

$$\int dk \left\{ \frac{\delta L_{\alpha, pk}^\beta}{\delta a_{-q}^\beta} \frac{\delta H}{\delta a_k^\alpha} + L_{\alpha, pk}^\beta \frac{\delta^2 H}{\delta a_{-q}^\beta \delta a_k^\alpha} - \frac{\delta L_{\beta, qk}^\alpha}{\delta a_{-p}^\alpha} \frac{\delta H}{\delta a_k^\beta} - L_{\beta, qk}^\alpha \frac{\delta^2 H}{\delta a_{-p}^\alpha \delta a_k^\beta} \right\} = 0, \quad (6.16)$$

$$\frac{\delta(L\Omega)_{\alpha\beta, pq}}{\delta a_{-k}^\gamma} + \frac{\delta(L\Omega)_{\gamma\alpha, kp}}{\delta a_{-q}^\beta} + \frac{\delta(L\Omega)_{\beta\gamma, qk}}{\delta a_{-p}^\alpha} = 0, \quad (6.17)$$

$$(L\Omega)_{\beta\alpha, qp} = -(L\Omega)_{\alpha\beta, pq} \quad (6.18)$$

where  $(L\Omega)_{\alpha\beta, pq} = \int dk L_{\alpha, pk}^\gamma \Omega_{\gamma\beta, -kq}$ .

In what follows we will consider nonlinear systems which are described by one real field. In this case a local and  $\mathcal{U}$ -independent symplectic form is of the form

$$\Omega_{(0) pq} = f_p \delta(p+q) \quad (6.19)$$

where  $f_p$  is an antisymmetric function  $f_{-p} = -f_p$  and  $\delta(p)$  is a Dirac-delta function. For the one-dimensional space ( $d=1$ ) without loss of generality one can choose

$$\Omega_{(0) pq} = -\frac{i}{p} \delta(p+q). \quad (6.20)$$

VII. Expansion over nonlinearity  
and recursion operator

The problem of calculation of the recursion operator, i.e. the problem of solution of equations (6.6)-(6.9) or (6.15)-(6.19) in functional derivatives, is difficult enough even in the simplest case of one field. One can simplify this problem if one restricts oneself to a certain special class of dynamical systems  $(\Omega, H)$  and solutions.

For this purpose we consider the translation-invariant systems which have a smooth behaviour at  $a_p \rightarrow 0$ , i.e. smoothly reduce to the linear system in this limit. So we assume that the Hamiltonian and the symplectic form of the translation-invariant system (6.1) are of the form

$$H = \sum_{n=2}^{\infty} \int dq_1 \dots dq_n \delta(q_1 + \dots + q_n) V_{(n)}(q_1 \dots q_n) a_{q_1} \dots a_{q_n}, \quad (7.1)$$

$$\Omega_{pq} = \sum_{n=0}^{\infty} \int dq_1 \dots dq_n \delta(p+q-q_1-\dots-q_n) \Omega_{(n)pq}(q_1 \dots q_n) a_{q_1} \dots a_{q_n} \quad (7.2)$$

where  $V_{(n)}(q_1 \dots q_n)$  are some functions which are completely symmetric on their variables and  $\Omega_{(n)pq}(q_1 \dots q_n)$  are functions which are symmetric on the variables  $q_1, \dots, q_n$ . For simplicity, we consider the case of one real field and successively use the momentum representation.

We will search for the recursion operator  $L_{pq}$  in the form of an "entire" function on  $a_p$  too:

$$L_{pq} = \sum_{n=0}^{\infty} \int dq_1 \dots dq_n \tilde{L}_{(n)pq}(q_1 \dots q_n) a_{q_1} \dots a_{q_n} \quad (7.3)$$

where  $\tilde{L}_{(n)pq}(q_1 \dots q_n)$  are functions which are completely symmetric over the variables  $q_1, \dots, q_n$ . The translation invariance gives certain restriction on the form of the functions

$\tilde{L}_{(n)pq}(q_1 \dots q_n)$ . Namely, taking into account that  $\{\vec{P}, a_q\} = -i\vec{q} a_q$  where  $\vec{P}$  is a total momentum, we have

$$\tilde{L}_{(n)pq}(q_1 \dots q_n) = \delta(p+q-q_1-\dots-q_n) L_{(n)pq}(q_1 \dots q_n) \quad (7.4)$$

where  $L_{(n)pq}(q_1 \dots q_n)$  are some functions.

Further, let us consider the case of constant symplectic form  $\Omega_{pq}$  and choose it as (6.19).

Let us substitute the expressions (7.1), (7.3), (7.4) and (6.19) into equations (6.15)-(6.18). The l.h.s. of these equations should be equal to zero in any order on  $a_p$ . Therefore, equations (6.15)-(6.18) are equivalent to the following functional equations for  $L_{(n)pq}(q_1 \dots q_n)$  and  $V_{(n)}(q_1 \dots q_n)$ :

$$\delta(p+q+k-k_1-\dots-k_n) \text{Sym} \sum_{(k_1, \dots, k_n)}^n \{ (n-m+1) L_{(m)p, \sum_{e=1}^m k_e - p}(k_1 \dots k_m) \times \\ \times L_{(n-m+1)q, k(\sum_{e=1}^m k_e - p, k_{m+1}, \dots, k_n)} + \quad (7.5)$$

$$+ (n-m+1) L_{(m) \sum_{e=1}^m k_e - k, k}(k_1 \dots k_m) L_{(n-m+1)p, k - \sum_{e=1}^m k_e}(-q, k_{m+1}, \dots, k_n) - (p \leftrightarrow q) \} = 0, \\ (n=0, 1, 2, 3, \dots)$$

$$\delta(p+q-q_1-\dots-q_n) \text{Sym} \sum_{(k_1, \dots, k_n)}^n \{ (m+1)(n-m+1) L_{(m+1)p, \sum_{e=1}^{n-m} k_e}(-q, k_1, \dots, k_m) \times \\ (7.6)$$

$$\times V_{(n-m+1) - \sum_{e=1}^{n-m} k_e, k_{m+1}, \dots, k_n} + (n-m+2)(n-m+1) L_{(m)p, p + \sum_{e=1}^m k_e}(k_1, \dots, k_m) \times$$

$$\times V_{(n-m+2) - q, -p + \sum_{e=1}^m k_e, k_{m+1}, \dots, k_n} - (p \leftrightarrow q) \} = 0, \quad (n=0, 1, 2, \dots)$$

$$\delta(p+q+k-k_1-\dots-k_n) \{ f_q L_{(n)pq}(-k, k_1, \dots, k_n) + f_p L_{(n)kp}(-q, k_1, \dots, k_n) + \\ + f_k L_{(n)qk}(-p, k_1, \dots, k_n) \} = 0, \quad (n=1, 2, 3, \dots) \quad (7.7)$$



$$\delta_{(p+q-k_1 \dots -k_n)} (f_q L_{(n)pq}(k_1 \dots k_n) + f_p L_{(n)qp}(k_1 \dots k_n)) = 0 \quad (7.8)$$

$$n = (0, 1, 2, \dots)$$

where  $\overset{\text{Sym}}{(k_1 \dots k_n)}$  denotes the complete symmetrization over the variables  $k_1, \dots, k_n$ .

The system of algebraic functional equations (7.5)-(7.8) is the complete system of equations for the calculations of all functions  $L_{(n)pq}(q_1 \dots q_n)$  which determine the recursion operator  $\mathcal{L}$ .

Here we present the simplest examples of equations (7.5), (7.6). Equations (7.5) with  $n = 0$  and  $n = 1$  are of the form

$$(L_{(0)p+q, -p-q} - L_{(0)q, -q}) L_{(1)p, -p-q(-q)} - (L_{(0)p+q, -p-q} - L_{(0)p, -p}) L_{(1)q, -p-q(-p)} = 0, \quad (7.9)$$

$$2(L_{(0)p, -p} - L_{(0)k, k}) L_{(2)q, k(-p, p+q+k)} - 2(L_{(0)q, -q} - L_{(0)k, k}) L_{(2)p, k(-q, p+q+k)} + L_{(1)p, q+k(p+q+k)} L_{(1)q, k(q+k)} + L_{(1)p+q, k(p+q+k)} L_{(1)p, -p-q(-q)} - (7.10)$$

$$-L_{(1)q, p+k(p+q+k)} L_{(1)p, k(p+k)} - L_{(1)p+q, k(p+q+k)} L_{(1)q, -p-q(-p)} = 0.$$

Equation (7.6) for  $n = 0$  gives

$$L_{(0)p, -p} V_{(2)-p, p} - L_{(0)-p, p} V_{(2)p, -p} = 0.$$

Since  $V_{(2)-p, p} = V_{(2)p, -p}$  then  $L_{(0)p, -p} = \varphi(p)$  where  $\varphi(p)$  is an arbitrary even function on  $p$  ( $\varphi(-p) = \varphi(p)$ ).

Further, equation (7.6) for  $n = 1$  is

$$L_{(1)p, -q-p(-p)} V_{(2)-q-p, q+p} + 3\varphi(p) V_{(3)-q, -p, p+q} + (7.11)$$

$$+ L_{(1)p, q(p+q)} V_{(2)-q, q} - (p \leftrightarrow q) = 0$$

and for  $n = 2$  it is of the form

$$3L_{(1)p, -p-q(-q)} V_{(3)-q-p, k, p+q+k} + 12\varphi(p) V_{(4)-q, -p, k, p+q-k} + 2L_{(1)p, -p-q+k(-q, k)} V_{(2)-q-p+k, q+p-k} + 2L_{(1)p, k(-q, p+q-k)} V_{(2)-k, k} + (7.12)$$

$$+ 3L_{(1)p, -p+k(k)} V_{(3)-q, -p+k, p+q-k} + 3L_{(1)p, q-k(p+q-k)} V_{(3)-q, q-k, k}$$

$$+ 2L_{(1)p, q(k, p+q-k)} V_{(2)-q, q} - (p \leftrightarrow q) = 0.$$

Equations (7.6) allows us to calculate all  $L_{(1)}$ ,  $L_{(2)}$ , ... , for given  $V_{(2)}$ ,  $V_{(3)}$ , ... . Let us start with equation (7.11). Taking into account the closeness and skewsymmetry conditions (7.7) and (7.8) it is not difficult to show that the relation (7.11) is equivalent to the following

$$L_{(1)pq}(p+q) = 3 \frac{\varphi(p) - \varphi(q)}{f_q(\omega(p+q) - \omega(p) - \omega(q))} V_{(2)-p, -q, p+q} \quad (7.13)$$

where  $\omega(p) \stackrel{\text{def}}{=} V_{(2)p, -p} / f_p$ . The function  $\omega(p)$  determines the dispersion law for the corresponding equation, i.e.

$$\frac{\partial a_p}{\partial t} = \omega(p) a_p + \dots$$

Analogously, using (7.13), we have from (7.12)

$$L_{(2)pq}(k, p+q-k) = 6 \frac{\varphi(p) - \varphi(q)}{f_q(\omega(p+q-k) + \omega(k) - \omega(p) - \omega(q))} V_{(3)-p, -q, k, p+q-k}$$

$$\begin{aligned}
& -\frac{g}{2} \frac{1}{f_g(\omega(p+q-k)+\omega(k)-\omega(p)-\omega(q))} \left( \frac{\varphi(p)-\varphi(q)}{f_{p+q}(\omega(p+q)-\omega(p)-\omega(q))} V_{(3) p, q, p+q}^* \right. \\
& \times V_{(3) q-p, k, p+q-k} - \frac{(\varphi(p)-\varphi(p-k)) V_{(3) p, p-k, k} V_{(3) q, -p+k, p+q-k}}{f_{k-p}(\omega(k)-\omega(p)-\omega(k-p))} + \\
& + \frac{(\varphi(q)-\varphi(q-k)) V_{(3) q, q-k, k} V_{(3) p, -q+k, p+q-k}}{f_{k-q}(\omega(k)-\omega(q)-\omega(k-q))} - \\
& \left. - \frac{(\varphi(p)-\varphi(q-k)) V_{(3) p, -q+k, p+q-k} V_{(3) q, q-k, k}}{f_{q-k}(\omega(p+q-k)-\omega(p)-\omega(q-k))} + \frac{(\varphi(q)-\varphi(p-k)) V_{(3) q, -p+k, p+q-k} V_{(3) p, p-k, k}}{f_{p-k}(\omega(p+q-k)-\omega(q)-\omega(p-k))} \right).
\end{aligned}
\tag{7.14}$$

In a similar manner one can derive, from (7.6), the formulae which express  $L_{(3)}$  via  $V_{(3)}$ ,  $V_{(4)}$ ,  $V_{(5)}$ , the function  $L_{(4)}$  via  $V_{(3)}$ ,  $V_{(4)}$ ,  $V_{(5)}$ ,  $V_{(6)}$  and so on.

Thus, for a given Hamiltonian, i.e. for given functions  $V_{(2)}$ ,  $V_{(3)}$ , ... and a certain fixed even function  $\varphi(p)$ , we easily calculate all functions  $L_{(1)pq(p+q)}$ ,  $L_{(2)pq(\dots)}$ , ... which determine the operator  $L_{pq}$ . In order that this operator  $L_{pq}$  be the recursion operator it should satisfy equations (7.5), (7.7) and (7.8). The fulfilment of (7.8) is obvious. By simple but tedious calculations, one can show that the expressions for  $L_{(1)pq(p+q)}$  and  $L_{(2)pq(\dots)}$ , given by the formulae (7.13), (7.14), indeed satisfy equations (7.9), (7.10) and the closeness conditions (7.7) for  $n = 1, 2$ .

Thus the formulae of the type (7.13), (7.14) allow one to calculate the recursion operator  $L_{pq}$  for given  $V_{(2)}$ ,  $V_{(3)}$ , ... . Emphasize that all functions  $V_{(2)}$ ,  $V_{(3)}$ , ... (i.e. the Hamiltonian of the equation) is arbitrary ones. So, any dynamical system (6.1) with any Hamiltonian of the form (7.1) possesses, at least, the formal strong recursion operator  $L_{pq}$ .

The existence of a formal strong recursion operator for any Hamiltonian system of the form (6.1), (7.1) becomes obvious

us if one takes into account the following three circumstances. The first one is: any system of equations (7.5)-(7.8) is invariant under the general transformations of "coordinates"  $a_p$  and, therefore, the existence of the recursion operator for this system is independent of the choice of variables  $a_p$ . Secondly, any nonlinear system with a Hamiltonian of the form (7.1) can be linearized by a suitable canonical transformation [32,33]. The third point is: any linear equation  $\frac{\partial a_p}{\partial t} = \omega(p)a_p$  with odd function  $\omega(p)$  possesses the recursion operator of the form  $L_{pq} = \varphi(p) \delta(p+q)$  where  $\varphi(p)$  is an arbitrary even function.

Indeed, let we have the dynamical system with Hamiltonian (7.1). Let us linearize this system (i.e. reduce the Hamiltonian  $H$  to the form  $H = \int dp_1 dp_2 \delta(p_1+p_2) V_{(2)p_1 p_2} b_{p_1} b_{p_2}$ ) by the canonical transformation  $a_p \rightarrow b_p$ :

$$a_p = b_p + \sum_{n=2}^{\infty} \int dp_2 \dots dp_n \delta(p-p_2-\dots-p_n) R_{(n)p(p_2, \dots, p_n)} b_{p_2} \dots b_{p_n}. \tag{7.15}$$

Using the condition of the canonical character of the transformation (7.15), i.e.

$$\{a_p, a_q\}_b = \int dk \frac{\delta a_p}{\delta b_k} \frac{1}{f_k} \frac{\delta a_q}{\delta b_{-k}} = \frac{1}{f_p} \delta(p+q) = \{a_p, a_q\}_a \tag{7.16}$$

we, in particular, have [32]

$$R_{(2)p+q(pq)} = -\frac{3}{2f_{p+q}(\omega(p+q)-\omega(p)-\omega(q))} V_{(3)p, q, -p-q}. \tag{7.17}$$

The linear equation  $\frac{\partial b_p}{\partial t} = \omega(p)b_p$ , which appears after this canonical transformation, possesses the recursion operator  $L_{pq}^{(linear)} = \varphi(p) \delta(p+q)$  where  $\varphi(p)$  is an arbitrary even

function. Let us now perform the inverse canonical transformation  $b_p \rightarrow a_p$  into the initial nonlinear system. The recursion operator is transformed under this transformation as follows:

$$\begin{aligned} L_{pq}^{(linear)} \rightarrow L_{pq} &= \int dk_1 dk_2 S_{-p, -k_1} L_{k_1 k_2}^{(linear)} \tilde{S}_{k_2, q} = \\ &= \int dk S_{-p, k} \varphi(k) \tilde{S}_{k, q} \end{aligned} \quad (7.18)$$

where  $S_{pq} \stackrel{\text{def}}{=} \delta b_p / \delta a_q$ ,  $\tilde{S}_{pq} = \delta a_q / \delta b_p$ . In particular, formula (7.18) gives

$$L_{(2)pq(p+q)} = -2(\varphi(q) - \varphi(p)) R_{(2)q}(-p, p+q). \quad (7.19)$$

Substitution of (7.17) into (7.19) gives exactly the expression (7.13). In a similar manner one can obtain the expression for  $L_{(2)pq}(k, p+q-k)$  of the form (7.14) and analogous formulae for  $L_{(3)}, L_{(4)}, \dots$ . So, formulae (7.18) yields the strong recursion operator for arbitrary initial nonlinear system with Hamiltonian (7.1).

### VIII. Regular recursion operator.

In the previous section it has been shown that any Hamiltonian equation possesses the formal strong recursion operator. However, in the general case such a recursion operator is a singular one due to the denominators of the forms

$$\omega(p+q) - \omega(p) - \omega(q), \quad \omega(p+q-k) + \omega(k) - \omega(p) - \omega(q),$$

etc., in the expressions of the type (7.13) and (7.14). Similar denominators are contained in the expressions for higher symplectic forms and Hamiltonians which are generated by the recursion operator. For example, for the symplectic form  $\Omega_{1pq} = (L\Omega_{(1)})_{pq} = L_{pq} f_{-q}$  we have

$$\Omega_{(2)pq(p+q)} = \frac{3(\varphi(q) - \varphi(p))}{\omega(p+q) - \omega(p) - \omega(q)} V_{(3)-p, -q, p+q}, \quad (8.1)$$

$$\Omega_{(2)pq}(k, p+q-k) = \frac{6(\varphi(q) - \varphi(p))}{\omega(p+q-k) + \omega(k) - \omega(p) - \omega(q)} V_{(4)-p, -q, k, p+q-k}^{\dagger} \dots$$

Using formulae (7.13) and (7.14), one can easily obtain the relations between the coefficient functions  $\tilde{V}_{(3)}$ ,  $\tilde{V}_{(4)}$ , and  $V_{(3)}$ ,  $V_{(4)}$  of the pair of Hamiltonians related by the recursion operator  $L$  ( $L \nabla H = \nabla \tilde{H}$ ). Equalizing the r.h.s. of the equalities (7.13) and (7.14), taken correspondingly for  $\tilde{V}_{(3)}$ ,  $\tilde{V}_{(4)}$  and  $V_{(3)}$ ,  $V_{(4)}$  and, taking into account an obvious equality  $\tilde{\omega}(p) = \varphi(p) \omega(p)$ , we obtain

$$\tilde{V}_{(3)-p, -q, p+q} = \frac{\varphi(p+q)\omega(p+q) - \varphi(p)\omega(p) - \varphi(q)\omega(q)}{\omega(p+q) - \omega(p) - \omega(q)} V_{(3)-p, -q, p+q}, \quad (8.2)$$

$$\tilde{V}_{(4)-p, -q, k, p+q-k} = \frac{\varphi(p+q-k)\omega(p+q-k) + \varphi(k)\omega(k) - \varphi(p)\omega(p) - \varphi(q)\omega(q)}{\omega(p+q-k) + \omega(k) - \omega(p) - \omega(q)} V_{(4)-p, -q, k, p+q-k}^{\dagger}$$

$$+ \frac{3}{4f_{pq}(\omega(p+q) - \omega(p) - \omega(q))} \left( \frac{\varphi(q+p)\omega(q+p) - \varphi(q+p-k)\omega(q+p-k) - \varphi(k)\omega(k)}{\omega(q+p) - \omega(q+p-k) - \omega(k)} \right)$$

$$\frac{\varphi(p+q-k)\omega(p+q-k) + \varphi(k)\omega(k) - \varphi(p)\omega(p) - \varphi(q)\omega(q)}{\omega(p+q-k) + \omega(k) - \omega(p) - \omega(q)} V_{(3)-p, -q, p+q} V_{(3)-q, -p, k, p+q-k}$$

$$-\frac{3}{4} \frac{\varphi(p) - \varphi(p-k)}{\varphi(p) - \varphi(q)} \left( \frac{\varphi(p+q-k)\omega(p+q-k) - \varphi(q)\omega(q) - \varphi(p-k)\omega(p-k)}{\omega(p+q-k) - \omega(q) - \omega(p-k)} \right) -$$

$$\frac{\varphi(p+q-k)\omega(p+q-k) + \varphi(k)\omega(k) - \varphi(p)\omega(p) - \varphi(q)\omega(q)}{\omega(p+q-k) + \omega(k) - \omega(p) - \omega(q)} \left( V_{(2), p, p-k, k} V_{(3), q, p+k, p+q-k}^+ \right)$$

$$+\frac{3}{4} \frac{\varphi(q) - \varphi(q-k)}{\varphi(p) - \varphi(q)} \left( \frac{\varphi(p+q-k)\omega(p+q-k) - \varphi(p)\omega(p) - \varphi(q-k)\omega(q-k)}{\omega(p+q-k) - \omega(p) - \omega(q-k)} \right) -$$

$$f_{k-q}(\omega(k) - \omega(q) - \omega(k-q))$$

$$-\frac{\varphi(p+q-k)\omega(p+q-k) + \varphi(k)\omega(k) - \varphi(p)\omega(p) - \varphi(q)\omega(q)}{\omega(p+q-k) + \omega(k) - \omega(p) - \omega(q)} \left( V_{(3), q, q-k, k} V_{(3), p, p+q-k, p+q-k}^- \right)$$

$$-\frac{3}{4} \frac{\varphi(p) - \varphi(q-k)}{\varphi(p) - \varphi(q)} \left( \frac{\varphi(k)\omega(k) - \varphi(q)\omega(q) - \varphi(k-q)\omega(k-q)}{\omega(k) - \omega(q) - \omega(k-q)} \right) \quad (8.3)$$

$$-\frac{\varphi(p+q-k)\omega(p+q-k) + \varphi(k)\omega(k) - \varphi(p)\omega(p) - \varphi(q)\omega(q)}{\omega(p+q-k) + \omega(k) - \omega(p) - \omega(q)} \left( V_{(3), p, q+k, p+q-k} V_{(3), q, q-k, k}^+ \right)$$

$$+\frac{3}{4} \frac{\varphi(q) - \varphi(p-k)}{\varphi(p) - \varphi(q)} \left( \frac{\varphi(k)\omega(k) - \varphi(p)\omega(p) - \varphi(k-p)\omega(k-p)}{\omega(k) - \omega(p) - \omega(k-p)} \right) -$$

$$f_{p-k}(\omega(p+q-k) - \omega(q) - \omega(p-k))$$

$$-\frac{\varphi(p+q-k)\omega(p+q-k) + \varphi(k)\omega(k) - \varphi(p)\omega(p) - \varphi(q)\omega(q)}{\omega(p+q-k) + \omega(k) - \omega(p) - \omega(q)} \left( V_{(3), q, p+k, p+q-k} V_{(3), p, p-k, k} \right)$$

Recall that all functionals  $H_n$  ( $L^n \nabla H = \nabla H_n$ ) are the integrals of motion for the initial Hamiltonian system (7.1). However, the presence of singularities (see e.g. formulae of the type (8.2) (8.3)) in the coefficient functions  $V_{(2)}$ ,  $V_{(3)}$ , ... of the higher integrals of motion  $H_1, H_2, \dots$  makes all these integrals poorly defined in the general case. In order that all these  $H_n$  be well defined functionals on  $\mathcal{A}_p$  it is necessary that the multiples of the form

$$\frac{\varphi(p+q)\omega(p+q) - \varphi(p)\omega(p) - \varphi(q)\omega(q)}{\omega(p+q) - \omega(p) - \omega(q)}, \quad (8.4a)$$

$$\frac{\varphi(p+q-k)\omega(p+q-k) + \varphi(k)\omega(k) - \varphi(p)\omega(p) - \varphi(q)\omega(q)}{\omega(p+q-k) + \omega(k) - \omega(p) - \omega(q)} \quad (8.4b)$$

would not contain the nonintegrable singularities.

We will refer to the recursion operator, which generates the family of well defined Hamiltonians  $H_n$  from the well defined initial Hamiltonian  $H$ , as the nonsingular recursion operator. It is clear that the demand of nonsingularity of the recursion operator leads to certain restrictions on the form of the functions  $\omega(p)$ ,  $V_{(2)}$ ,  $V_{(3)}$ , ... and extract some subclass of equations from all equations of the form (7.1).

The stronger restriction on the Hamiltonian (i.e. on the functions  $V_{(2)}$ ,  $V_{(3)}$ ,  $V_{(4)}$ , ...) appears if one demands that all functions  $V_{(2)}$ ,  $V_{(3)}$ ,  $V_{(4)}$ , ... would not have singularities at all. We will refer to the recursion operator which produces such a family of Hamiltonians as a regular recursion operator.

As we shall see, the properties of recursion operators crucially depends on the dimensionality  $d$  of the space. Let us consider subsequently the cases  $d = 1$ ,  $d = 2$ ,  $d \geq 3$ .

In the one-dimensional space the multiple (8.4) is the simplest one. For  $d = 1$  we have  $\omega(p) = \sum_{n=1}^{\infty} \alpha_n p^{2n+1}$  and  $\varphi(p) = \sum_{n=0}^{\infty} \beta_n p^{2n}$ . It is not difficult to show that for any  $\omega(p)$  and  $\varphi(p)$  the expressions (8.4) are polynomials on  $p$ ,  $q$  and  $p, q, k$ . For example, for  $\omega = p^3$ ,  $\varphi = p^2$  one has

$$\frac{\varphi(p+q)\omega(p+q) - \varphi(p)\omega(p) - \varphi(q)\omega(q)}{\omega(p+q) - \omega(p) - \omega(q)} = \frac{5pq(p+q)(p^2+pq+q^2)}{3pq(p+q)} = \frac{5}{3}(p^2+pq+q^2)$$

$$\frac{\varphi(p+q-k)\omega(p+q-k) + \varphi(k)\omega(k) - \varphi(p)\omega(p) - \varphi(q)\omega(q)}{\omega(p+q-k) + \omega(k) - \omega(p) - \omega(q)} = \quad (8.5)$$

$$= \frac{5(p+q)(q-k)(p-k)(p^2+q^2+k^2-pk+pq-qk)}{3(p+q)(q-k)(p-k)} =$$

$$= \frac{5}{3}(p^2+q^2+k^2-pk+pq-qk)$$

Thus for any  $\omega(p)$  and  $\varphi(p)$  the function  $\bar{V}_{(3)-p,-q,k,p+q}$  has no singularities.

The expressions in the round brackets in (8.4) have no singularities, too. Let us choose  $f_p = \text{const } p^{-2\gamma-1}$  ( $\gamma > 0$ ). It is not difficult to see that by virtue of the multipliers in front of the round brackets in (8.4) the expression for

$\bar{V}_{(4)-p,-q,k,p+q-k}$  has the poles at  $p+q=0$ ,  $p-q=0$ ,  $p=0$  and  $q=0$ . However, it is easy to check that the residues of  $\bar{V}_{(4)-p,-q,k,p+q-k}$  in these poles are equal to zero independently of the form of  $V_{(3)} \dots$

So the functions  $\bar{V}_{(3)}$ ,  $\bar{V}_{(4)}$  have no singularities for any  $\omega(p)$  and  $\varphi(p)$ . One can, however, show that the function  $\bar{V}_{(5)}$  has no such a property. Analogously,  $\bar{V}_{(6)}$ ,  $\bar{V}_{(7)}$ , ... have singularities for general  $V_{(3)}$ ,  $V_{(4)}$ , ... . The requirement for the absence of singularities in the expressions for all the functions  $\bar{V}_{(5)}$ ,  $\bar{V}_{(6)}$ , ... leads to a certain system of equations for  $V_{(3)}$ ,  $V_{(4)}$ ,  $V_{(5)}$ , ... . If these equations are satisfied, then the considered system possesses the regular recursion operator. In virtue of the cumbersome form of these equations, we omit them here.

In the general case all the functions  $L_{(1)}$ ,  $L_{(2)}$ ,  $L_{(3)}$ , ... , which determine the recursion operator, are not equal to

zero, i.e. the recursion operator is the entire functional on  $\mathcal{A}$ . However, it may occur that this infinite series is interrupted on some (N-th) term. We will refer to such a recursion operator as the N-linear recursion operator. It is clear that the requirement of the N-linearity of the recursion operator, i.e. the requirement  $L_{(N+1)} = L_{(N+2)} = \dots = 0$  leads to strong restrictions on the form of the functions  $V_{(2)}$ ,  $V_{(3)}$ ,  $V_{(4)}$ , ... . For example, in order that the recursion operator  $L$  be linear on  $\mathcal{A}$  (i.e.  $L_{(2)} = L_{(3)} = \dots = 0$ ) it is necessary that the r.h.s. of (7.14) should be equal to zero, i.e.

$$\frac{4}{3}(\varphi(p) - \varphi(q)) V_{(4)-p,-q,k,p+q-k} -$$

$$- \frac{\varphi(p) - \varphi(q)}{f_{p+q}(\omega(p+q) - \omega(p) - \omega(q))} V_{(3)-p,-q,p+q} V_{(3)-q-p,k,p+q-k} +$$

$$+ \frac{1}{f_{p-k}} \left( \frac{\varphi(p) - \varphi(p-k)}{\omega(k) - \omega(p) - \omega(k-p)} + \frac{\varphi(q) - \varphi(p-k)}{\omega(p+q-k) - \omega(q) - \omega(p-k)} \right) V_{(3)-p,p-k,k} V_{(3)-q,-p+k,p+q-k} \quad (8.6)$$

$$- \frac{1}{f_{q-k}} \left( \frac{\varphi(q) - \varphi(q-k)}{\omega(k) - \omega(q) - \omega(k-q)} + \frac{\varphi(p) - \varphi(q-k)}{\omega(p+q-k) - \omega(p) - \omega(q-k)} \right) \times$$

$$\times V_{(3)-q,q-k,k} V_{(3)-p,-q+k,p+q-k} = 0.$$

In addition to (8.6), the conditions on  $V_{(2)}$ ,  $V_{(3)}$ ,  $V_{(4)}$ , ... , which mean that  $L_{(2)} = L_{(4)} = \dots = 0$  should be satisfied.

In the case when Hamiltonian  $H$  is cubic on  $\mathcal{A}$ , i.e. when  $V_{(4)} = V_{(5)} = \dots = 0$ , the whole system of equations is reduced to the only equation (8.6) with  $V_{(4)} = 0$ . Let us consider this equation in the simplest case  $f_p = -i/p$ ,  $\varphi(p) = \alpha + \beta p^2$ ,  $\omega(p) = \text{const } p^3$ . After the trivial transformations equa-

tion (8.6) reduces to the following:

$$\begin{aligned} & (q-p) V_{(\beta)-p,-q,p+q} V_{(\beta)-q-p,k,p+q-k} + \\ & + (k-q) V_{(\beta)-p,p-k,k} V_{(\beta)-q,-p+k,p+q-k} + \\ & + (p-k) V_{(\beta)-q,q-k,k} V_{(\beta)-p,-q+k,p+q-k} = 0. \end{aligned} \quad (8.7)$$

It is not difficult to verify that equation (8.7) has the only solution  $V_{(\beta)} = \text{const}$  within the class of polynomial  $V_{(\beta)}$ , i.e. within the class of local Hamiltonians  $H$ . Indeed, putting in (8.7)  $p=k=0$  one gets  $q(V_{(\beta)0,-q,q} - V_{(\beta)0,0,0}) V_{(\beta)-q,0,q} = 0$ . Therefore  $V_{(\beta)0,-q,q} = V_{(\beta)0,0,0} = \text{const}$ .

Thus, among the nonlinear equations with threelinear local Hamiltonians and  $\omega(p) = p^3$  only the equation with  $V_{(\beta)} = \gamma/3 = \text{const}$ , i.e. the KdV equation, possesses the linear recursion operator. It is of the form

$$L_{pq} = (\alpha + \beta p^2) \delta(p+q) + \gamma \frac{p-q}{p} a_{p+q} \quad (8.8)$$

where  $\alpha, \beta, \gamma$  are arbitrary constants. For  $\alpha = 0, \beta = -1$  and  $\gamma = -2$  the operator (8.8) is the recursion operator (1.2) in the momentum representation.

Another example is the bilinear recursion operator which corresponds to the equation with  $\omega(p) = -ip^3, V_{(4)} = \gamma/6 = \text{const}$  and  $V_{(5)} = V_{(6)} = \dots = 0$ . In this case,

$$\begin{aligned} L_{(1)pq} &= (\alpha + \beta p^2) \delta(p+q), \\ L_{(2)pq(k,p+q-k)} &= \gamma \frac{(p-q)q}{pk - k(p+q-k)} \\ L_{(3)} &= L_{(4)} = L_{(5)} = \dots = 0 \end{aligned} \quad (8.9)$$

i.e.

$$L_{pq} = (\alpha + \beta p^2) \delta(p+q) + \gamma \int_{-\infty}^{+\infty} dk \frac{q}{p-k} a_k a_{p+q-k}$$

where  $\alpha, \beta, \gamma$  are arbitrary constants. For  $\alpha = 0, \beta = -1, \gamma = -1$  the operator (8.9) is the recursion operator for the modified KdV equation (see [11]) in momentum representation. Note that the operators (8.8) and (8.9) in coordinate representation have been calculated by another technique in [28].

Thus, in the one-dimensional space there exist Hamiltonian equations which possess the regular recursion operator. For a certain subclass of these equations the recursion operator is the polynomial of the finite order on the field  $\mathcal{A}$ .

The situation changes dramatically when we transit to the two-dimensional space. It is connected with the circumstance that the expressions of the form (8.4) have no singularities only for certain special functions  $\omega(p)$  and  $\varphi(p)$ . Let us first consider the expression (8.4a). For the absence of singularities in such expressions it is necessary that the numerator be equal to zero on the same manifold  $\Gamma^{1,2}$  as the denominator. This means that the dispersion law  $\omega(p)$  should be a degenerative one with respect to the decay process  $1 \rightarrow 2 + 3$  (for the degenerative dispersion laws see [34,35]) and dispersion law  $\varphi(p)\omega(p)$  should belong to the class of degenerative dispersion laws associated with  $\omega(p)$ . The second condition can be easily fulfilled if for the degenerative dispersion law  $\omega(p)$  one chooses  $\varphi(p) = \tilde{\omega}(p)/\omega(p)$ , where  $\tilde{\omega}(p)$  is any dispersion law associated with given dispersion law  $\omega(p)$ . A wide class of degenerative dispersion laws has been described in [34,35].

In the description of the dispersion laws the dimension of the manifold  $\Gamma^{n,m}$  which is defined by the equations

$$\begin{aligned} p_1 + \dots + p_n &= p_{n+1} + \dots + p_{n+m}, \omega(p_1) + \dots + \omega(p_n) = \\ &= \omega(p_{n+1}) + \dots + \omega(p_{n+m}) \end{aligned}$$

plays an important role. If  $\dim \Gamma^{1,2}$  and  $\dim \Gamma^{2,2}$  are less than a maximal one (i.e.  $\dim \Gamma^{1,2} < 2d-1, \dim \Gamma^{2,2} < 3d-1$ ), then the expressions  $(\omega(p+q) - \omega(p) - \omega(q))^{-1}$  and  $(\omega(p+q-k) + \omega(k) - \omega(p) - \omega(q))^{-1}$  may have the integrable

singularities. For such  $\omega(p)$  (with  $\dim \Gamma^{n,m} < \max \dim \Gamma^{n,m}$ ) the nonsingular recursion operator may exist.

If  $\dim \Gamma^{n,m}$  is a maximal one, then the corresponding nonlinear equation does not possess the nonsingular recursion operator. Indeed in this case, in virtue of the theorem proved in [35], the expressions of the form (8.4b) have nonintegrable singularities. The multipliers of the form  $\frac{\varphi(p) - \varphi(p-k)}{\varphi(p) - \varphi(q)} (f_{k-p}(\omega(k) - \omega(p) - \omega(k-p)))^{-1}$  in front of the round brackets in (8.4) have the nonintegrable singularities too. Moreover, the nonintegrable singularity is contained in the symplectic form  $\Omega_{2,1} p q (p+q)$  (see formula (8.1)).

Thus, the nonlinear equations in the two-dimensional space, which describe the nontrivial scattering of  $n$  waves into  $m$  waves ( $n \neq m$ ) with  $\max \dim \Gamma^{n,m}$ , does not possess the nonsingular recursion operator. In particular, the well-known Kadomtsev-Petviashvili equation [1], for which  $\dim \Gamma^{1,2} = 3$  and  $V(s) = \text{const}$ , has no the nonsingular recursion operator.

An analogous situation takes place for three and higher-dimensional spaces ( $d \geq 3$ ). Since for  $d \geq 3$  there exist no degenerative dispersion laws with  $\max \dim \Gamma^{n,m}$  [35], the nonlinear equations with  $\max \dim \Gamma^{n,m}$  do not possess, nonsingular recursion operators. Only the equations with  $\dim \Gamma^{n,m} < \max \dim \Gamma^{n,m}$  may have the nonsingular recursion operator.

So we see that the regular recursion operator is a pure one-dimensional phenomenon. The proposed method of expansion over the fields (i.e. the perturbation theory method) seems to be adequate for an analysis of the problem of existence or non-existence of the nonsingular recursion operator in the multi-dimensional spaces. All the results of sections 7 and 8 can be generalized to the case of the systems of equations (6.1).

## References

1. Zakharov V.E., Manakov S.V., Novikov S.P., Pitaevski L.P., Theory of solitons. Method of the Inverse Problem, Nauka, Moscow 1980.
2. Solitons. Topics in Current Physics, Vol. 17, Bullough R., Caudrey P. (eds.), Heidelberg: Springer 1980.
3. Ablowitz M.J., Segur H., Solitons and Inverse Scattering Transform, SIAM, Philadelphia 1981.
4. Gardner C.S., Green J.H., Kruskal H.D., Miura R.M., Comm. Pure Appl. Math., 27, 97 (1974).
5. Olver P.J., J. Math. Phys., 18, 1212-1215 (1977).
6. Gardner C.S., J. Math. Phys., 12, 1548 (1971).
7. Zakharov V.E., Faddeev L.D., Funct. Analysis and its Appl., 5, N 4, 18 (1971).
8. Magri F., J. Math. Phys., 19, 1156 (1978).
9. Kulish P.P., Reiman A.G., Notes of LOMI scientific seminars, 77, 134-147 (1978).
10. Ablowitz M.J., Kaup D.J., Newell A.C., Segur H., Stud. Appl. Math., 53, 249-315 (1974).
11. Flaschka H., Newell A.C., Lecture Notes in Physics, 38, 355-440 (1975).
12. Newell A.C., Proc. Roy. Soc. London, A365, 283-311 (1979).
13. Gerdjikov V.S., Ivanov M.I., Kulish P.P., Teor. Mat. Fyz., 44, 342 (1980).
14. Konopelchenko B.G., Phys. Lett., 75A, 447-450 (1980), 79A, 39-43 (1980); 95B, 83-89 (1980); 100B, 254-260 (1981); 108B, 26-32 (1982).
15. Kulish P.P., Notes of LOMI scientific seminars, 96, 105-112 (1980).
16. Konopelchenko B.G., J. Phys. A: Math. Gen., 14, 1237-1259 (1981); 14, 3125-3143 (1981).
17. Alonso L.M., J. Math. Phys., 21 2342-2349 (1980); Lett. Math. Phys., 4, 215 (1980).

18. Gerdjikov V.S., Kulish P.P., *Physica* 3D, 549-564 (1981).
19. Konopelchenko B.G., *Funct. Analysis and its Appl.*, 16, N 3, 63-65 (1982); *Lett. Math. Phys.*, 6, 309-314 (1982); *Physica D* (to be published).
20. Konopelchenko B.G., *Fortschritte der Physik*, 31, 253 (1983).
21. Dubrovsky V.G., Konopelchenko B.G., *Fortschritte der Physik* 32, N 4 (1984); Preprint Institute of Nuclear Physics 82-09 (1982); Konopelchenko B.G., Dubrovsky V.G., *Annals of Physics* (1984) (to appear), preprint Institute of Nuclear Physics 83-026 (1983).
22. Gadjiev I.T., Gerdjikov V.S., Ivanov M.I., *Notes of LOMI scientific seminars*, 120, 55-68 (1982).
23. Konopelchenko B.G., *Commun. Math. Phys.*, 87, 105-125 (1982); 88, 531-549 (1983).
24. Adler M., *Inventiones Math.*, 50, 219-248 (1979).
25. Reiman A.G., Semenov-Tyan-Shansky M.A., *Funct. Analysis and its Appl.*, 14, N 2, 77-78 (1980).
26. Gelfand I.M., Dorphman I.Ya., *Funct. Anal., and its Appl.*, 13, N 4, 13-30 (1979); 14, N 3, 71-74 (1980).
27. Fuchssteiner B., *Nonlinear Anal., Theory, Methods and Appl.*, 3, 842-862 (1979).  
Fuchssteiner B., *Prog. Theor. Phys.*, 65, 861-876 (1981); 68, 1082-1104 (1982).
28. Fuchssteiner B., Fokas A.S., *Physica* 4D, 47-66 (1981).
29. Fokas A.S., Anderson R.L., *J. Math. Phys.*, 23, 1066-1073 (1982).
30. Magri F., *Lecture Notes in Physics*, 120, 233-263 (1980).
31. Arnold V.I., *Mathematical Methods of Classical Mechanics*, Nauka, Moscow, 1974.
32. Zakharov V.E., *Izvestiya VUZov. Radiofizika*, 17, 431-453 (1974).

33. Zakharov V.E., Kuznetsov E.A., preprint of Institute of automatics and electrometrii, N 186, Novosibirsk (1982).
34. Zakharov V.E., Schulman E.I., *Physica* 1D, 192-202 (1980).
35. Zakharov V.E., *Lecture Notes in Physics*, 153, 190-216 (1982).



В.Е.Захаров, Б.Г.Конопельченко

К ТЕОРИИ РЕКУРСИОННОГО ОПЕРАТОРА

Препринт  
№ 83-151

Работа поступила - 9 июня 1983 г.

---

Ответственный за выпуск - С.Г.Попов

Подписано к печати 14.XI-1983 г. МН 03396

Формат бумаги 60x90 1/16 Усл.3,0 печ.л., 2,4 учетно-изд.л.

Тираж 290 экз. Бесплатно. Заказ № 151.

---

Ротапринт ИЯФ СО АН СССР, г.Новосибирск, 90