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BACKLUND-CALOGERO GROUP AND GENERAL FORM OF INTEGRABLE EQUATIONS FOR THE TWO-DIMENSIONAL GELFAND-DIKIJ-ZAKHAROV-SHABAT PROBLEM. BILOCAL APPROACH

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A b s t r a c t

General form of the integrable equations and their Backlund transformations connected with the general two-dimensional Gelfand-Dikij-Zakharov-Shabat spectral problem is found within the framework of generalized AKNS method. Bilocal tensor product of the solutions of the spectral problem is used successively that essentially simplifies the calculations of recursion operators. Transformation properties of the integrable equations and Backlund transformations under the gauge group are discussed.

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I. Introduction

The inverse scattering transform method allows to investigate in details a wide class of nonlinear differential equations (see e.g. [1-3]). Various versions of this method are intensively developed in the present time. The generalized AKNS method, proposed for the first time in [4] and then developed in [5-16], looks like the most attractive one from the point of view of description of the integrable equations and analysis of their group-theoretical properties. So called recursion operator plays a central role in AKNS method. A calculation of recursion operator in the explicit form is the main problem of this method. It succeded to do for a wide class of one dimensional spectral problems [6-16]. A concept of recursion operator has been also generalized to certain twodimensional spectral problems [17, 18].

In the present paper we consider the twodimensional Gelfand-Dikij-Zakharov-Shabat problem

$$\frac{\partial^N\psi}{\partial x^N}+V_{N-1}(x,y,t)\frac{\partial^{N-1}\psi}{\partial x^{N-1}}+\ldots+V_1(x,y,t)\frac{\partial\psi}{\partial x}+V_0(x,y,t)\psi+\frac{\partial\psi}{\partial y}=0 \qquad (1.1)$$

where N is arbitrary integer and $V_0(x,y,t),\ldots,V_{N-1}(x,y,t)$ are scalar functions such that $\lim\limits_{\sqrt{x^2+y^2}\to\infty}V_k(x,y,t)\longrightarrow 0\ (k=0,1,\ldots,N-1)$ in the frame of generalized AKNS method. The problem (1.1) for $V_{N-1}=0$ has been considered already in the framework of AKNS method in [18]. In the present paper we will work in the frame of this method too, but we will use essentially another technique. We will use successively the bilocal tensor product $\hat F'(x,\tilde y,t)\otimes \check F(x,y,t)$ of the solutions of the problem (1.1) and it's adjoint problem. It allows essentially simplify the calculations of the recursion operators and makes the whole twodimensional version of

AKNS method much more transparent and clear. It seems this bilocal formulation is more adequate to the twodimensional nature of the problem.

We construct the infinite-dimensional abelian Backlund-Calogero group of general Backlund transformations connected with the problem (1.1) and find the general form of nonlinear equations integrable by (1.1). We construct and use the recursion operators which act on the whole N-dimensional space and do not impose any gauge conditions on potentials $V_0, \dots, V_{N-1}$.

In the paper we consider also the transformati… properties of the general equations integrable by (1.1) and t… r Backlund transformations under the gauge transformations whic… conserve the problem (1.1). Manifestly gauge invariant formulation of the integrable equations and Backlund transformations is given.

The paper is organize… as follows. In the second section we obtain certain important equations for bilocal quantity $\hat{F}'(x,\tilde{y},t) \otimes \check{F}(x,y,t)$. Recursion operators are calculated in section 3. Backlund-Calogero group and general form of the integrable equations are found in section 4. In section 5 we discuss the gauge invariance of the integrable equations and Backlund transformations. In sections 6 and 7 we present the examples of integrable equations and Backlund transformations in the simplest cases N = 2 and N = 3.

II. Adjoint representation and some important relations

Firstly we represent the problem (1.1) in the well known matrix form

$$\frac{\partial \hat{F}}{\partial x} + A \frac{\partial \hat{F}}{\partial y} + P(x,y,t)\hat{F} = 0 \tag{2.1}$$

where

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & -1 \\ V_0 & V_1 & V_2 & \dots & V_{N-1} \end{pmatrix} . \tag{2.2}$$

The adjoint problem is

$$\frac{\partial \check{F}}{\partial x} + \frac{\partial \check{F}}{\partial y} A - \check{F} P(x,y,t) = 0 . \tag{2.3}$$

Let us introduce now the tensor product of the solutions of the problems (2.1) and (2.3)

$$\varphi_{k\ell}^{in}(x,\tilde{y},y,t) \stackrel{def}{=} \hat{F}'_{kn}(x,\tilde{y},t)\check{F}_{i\ell}(x,y,t) \tag{2.4}$$
$$i,k,\ell,n = 1,\dots,N .$$

The quantity $\varphi_{k\ell}^{in}(x,\tilde{y},y,t)$ is transformed under the adjoint representation of the group which is defined by the problem (1.1). An important role of this quantity in the inverse scattering transform method has been discussed in [19]. Emphasize that the solutions $\hat{F}'(x,\tilde{y},t)$ and $\check{F}(x,y,t)$ in (2.4) are in the different on the second spatial variable points $\tilde{y}$, y and correspond to different potentials $P'(x,\tilde{y},t)$ and $P(x,y,t)$. A touch in (2.4) and below will denote the quantities which correspond to the potential $P'(x,\tilde{y},t)$ (e.g. F').

In the present paper we will successively use the quantity $\varphi_{k\ell}^{in}(x,\tilde{y},y,t)$. The bilocal nature of this quantity will play an important role.

Using (2.1) and (2.3) it is easy to obtain the equation for $\varphi_{k\ell}^{in}(x,\tilde{y},y,t)$. It is of the form

$$\frac{\partial \varphi^{in}(x,\tilde{y},y,t)}{\partial x} + A \frac{\partial \varphi^{in}(x,\tilde{y},y,t)}{\partial \tilde{y}} + \frac{\partial \varphi^{in}(x,\tilde{y},y,t)}{\partial y} A + \tag{2.5}$$
$$+ P'(x,\tilde{y},t)\varphi^{in}(x,\tilde{y},y,t) - \varphi^{in}(x,\tilde{y},y,t)P(x,y,t) = 0 .$$

Equation (2.5) is the main equation for the construction of general Backlund transformations connected with the problem (1.1).

Let us firstly obtain certain important preliminary relation. Introduce for this purpose a matrix operator $B(-\partial_{\tilde{y}},t)$ given by

$$B(-\partial_{\tilde{y}},t) = \sum_{k=0}^{N-1} B_k(-\partial_{\tilde{y}},t)\left(-A\partial_{\tilde{y}} + P_\infty\right)^k = \sum_{k=0}^{N-1} B_k(-\partial_{\tilde{y}},t)\left(-(P_\infty^T)^{N-k}\partial_{\tilde{y}} + P_\infty^k\right) \tag{2.6}$$

where $P_\infty = \lim\limits_{\sqrt{x^2+y^2}\to\infty} P(x,y,t)$ and $B_k(-\partial_{\tilde y},t)$ $(k=0,1,\dots,N-1)$ are arbitrary scalar functions. Let us multiple (2.5) from the left by the matrix $B(-\partial_{\tilde y},t)$, take trace and integrate the obtained expression with Dirac delta function $\delta(\tilde y - y)$ over x, $\tilde y$ and y. Taking into account the commutativity of $B(-\partial_{\tilde y},t)$ with $-A\partial_{\tilde y} + P_\infty$ and using (2.4), one can represent the result in the form

$$\int_{-\infty}^{+\infty} dx\,dy\,\{\partial_x A_x^{in} + \partial_y A_y^{in}\} + \int_{-\infty}^{+\infty} dx\,dy\,d\tilde y\,\delta(y-\tilde y)\,\cdot \tag{2.7}$$

$$\cdot\, tr\{B(-\partial_{\tilde y},t)\tilde P'(x,\tilde y,t)\varphi^{in}(x,\tilde y,y,t) - B(-\partial_{\tilde y},t)\varphi^{in}(x,\tilde y,y,t)\tilde P(x,y,t)\} = 0$$

where $\tilde P = P + P_\infty$ and $A_x^{in} = (\check F(x,y,t)B(-\partial_y,t)\hat F'(x,y,t))_{in}$, $A_y^{in} = (\check F(x,y,t)AB(-\partial_y,t)\hat F'(x,y,t))_{in}$.

The first term in (2.7) through Gauss theorem can be transformed into integral over surface. We will consider only the functions $B_k(-\partial_{\tilde y},t)$ entire on $-\partial_{\tilde y}$, i.e.

$$B_k(-\partial_{\tilde y},t) = \sum_{n=0}^{\infty} b_{kn}(t)(-\partial_{\tilde y})^n \tag{2.8}$$

where $b_{kn}(t)$ are arbitrary functions. We assume also that $V_k(x,y,t)\to 0$ at $\sqrt{x^2+y^2}\to\infty$ so fast that there exist the solutions $\hat F'$ and $\check F$ of (2.1) and (2.3) which decrease at $\sqrt{x^2+y^2}\to\infty$ faster than $1/(x^2+y^2)^{\frac{1}{2}+\varepsilon}$, $\varepsilon>0$.

For such solutions $\hat F'$, $\check F$ and functions B_k of the form (2.8) the integral over infinite surface and therefore the first term in (2.7) is equal to zero. At the result we have the following important equation

$$\int_{-\infty}^{+\infty} dx\,d\tilde y\,dy\,\delta(y-\tilde y)\,tr\{B(-\partial_{\tilde y},t)\tilde P'(x,\tilde y,t)\,\varphi^{in}(x,\tilde y,y,t) - B(-\partial_{\tilde y},t)\varphi^{in}(x,\tilde y,y,t)\tilde P(x,y,t)\} = 0. \tag{2.9}$$

Using the direct scattering problem for (2.1) and (2.3) one can show [18] that the equation (2.9) is valid for some other class of solutions $\hat F'$, $\check F$ too.

Introduce follow to [18] the matrices-solutions $\hat F_\lambda^{\pm}(x,y,t)$ of the problem (2.1) given by their asymptotics

$$\hat F_\lambda^{\pm}(x,y,t) \xrightarrow[x\to\pm\infty]{} (2\pi i)^{-\frac{1}{2}}\lambda^{\frac{N-1}{2}} D(\lambda)\exp(-\lambda^N y + \bar A(\lambda)x) \tag{2.10}$$

where $\lambda\in\mathbb{R}$; $\bar A(\lambda)$ is diagonal matrix: $\bar A_{ik} = \lambda q^{i-1}\delta_{ik}$, $(i,k=1,\dots,N)$; $q = \exp\frac{2\pi i}{N}$, δ_{ik} is Kronecer symbol and $D_{ik} = (\lambda q^{k-1})^{i-1}$, $(i,k=1,\dots,N)$. The numbers λq^{i-1} are eigenvalues of the matrix $\tilde A = \lambda^N A + P_\infty$ and $A = D(\lambda)\bar A D^{-1}(\lambda)$. For adjoint problem (2.3) we introduce the solutions $\check F_\lambda^{\pm}(x,y,t)$:

$$\check F_\lambda^{\pm}(x,y,t) \xrightarrow[x\to\pm\infty]{} (2\pi i)^{-\frac{1}{2}}\lambda^{\frac{N-1}{2}}\exp(\lambda^N y - \bar A(\lambda)x)D^{-1}(\lambda) \tag{2.11}$$

Let us choose the function $\varphi^{in}(x,\tilde y,y,t)$ in (2.7) as $\overset{++}{\varphi}(x,\tilde y,y,t) = \hat F_\lambda^{+\prime}(x,\tilde y,t)\otimes\check F_{\tilde\lambda}^{+}(x,y,t)$ where the solutions $\hat F_\lambda^{+\prime}(x,\tilde y,t)$ and $\check F_{\tilde\lambda}^{+}(x,y,t)$ are given by (2.10) and (2.11). It is not difficult to show, assuming $\int_{-\infty}^{+\infty} dy\frac{\partial}{\partial y}(\dots) = 0$ and using the calculations analogous to those given in [18], that $\int_{-\infty}^{+\infty} dx\,dy\,\partial_x A_x^{in} = 0$ for $i\neq n$. As a result we again obtain (2.9). Correspondingly the scattering matrix $S(\tilde\lambda,\lambda,t)$, which is defined by the formula $\hat F_\lambda^{+}(x,y,t) = \int d\tilde\lambda\,\hat F_{\tilde\lambda}^{-}(x,y,t)S(\tilde\lambda,\lambda,t)$ should transform in the following way

$$S(\tilde\lambda,\lambda,t)\longrightarrow S'(\tilde\lambda,\lambda,t) = B^{-1}(\tilde\lambda,t)S(\tilde\lambda,\lambda,t)C(\lambda,t) \tag{2.12}$$

where $B_{ik}(\tilde\lambda,t) = \delta_{ik}B_k(\tilde\lambda,t)$ and $C(\lambda,t)$ is an arbitrary diagonal matrix.

Rewrite the relation (2.9) in a form which is more convenient for our further calculations. Firstly we change in (2.9) $B(-\partial_{\tilde y},t)$ on $B(\partial_y,t)$. It is possible due to the equality $\int_{-\infty}^{+\infty} dy\,d\tilde y\,\delta(y-\tilde y)(\partial_y+\partial_{\tilde y})(\dots) = \int_{-\infty}^{+\infty} dy\frac{\partial}{\partial y}(\dots) = 0$. Then substituting the expressions (2.6) for $B(\partial_y,t)$ and $B(-\partial_{\tilde y},t)$ into (2.9) and using (2.2) and (2.8), we finally obtain

$$\begin{aligned}\int_{-\infty}^{+\infty} dx\,dy\,d\tilde y\,\delta(y-\tilde y)\sum_{k=0}^{N-1}\sum_{n=0}^{\infty} b_{kn}(t)\,tr\{\partial_y^n\varphi^{il}_{\Delta_{N-k}}(x,\tilde y,y,t)P_\infty^k\tilde P'(x,\tilde y,t) - \\ -(-1)^{n+1}(P_\infty^T)^{N-k}\partial_{\tilde y}^{n+1}\varphi^{il}_{\Delta_N}(x,\tilde y,y,t)\tilde P(x,y,t) - (-1)^n P_\infty^k\partial_{\tilde y}^n\varphi^{il}_{\Delta_N}(x,\tilde y,y,t)\tilde P(x,y,t)\} = \\ = 0\end{aligned} \tag{2.13}$$

where Δ_k denotes a projection of matrix φ onto k-th column;

$$(\varphi_{\Delta_K})_{mn} \overset{def}{=} \delta_{nK}\, \varphi_{mK} \quad .$$

If one perform the integration over $\tilde{y}$ in (2.13), the integrand become the local one on y and one can produce further calculations analogously to [18].

But it is more convenient do not perform the integration over $\tilde{y}$ in (2.13) up to the very end and to work with nonlocal expressions as long as possible. The calculation of recursion operator and construction of Backlund transformations and integrable equations become much more simple and transparent in such bilocal approach.

III. Recursion operators

For further transformation of the expression (2.13) it is necessary to establish the relations between the quantities $\partial_{\tilde{y}}^{n}\,\varphi(x,\tilde{y},y,t)\big|_{\tilde{y}=y}$ and $\partial_{\tilde{y}}^{n}\,\varphi(x,\tilde{y},y,t)\big|_{\tilde{y}=y}$ with different n, i.e. to calculate the recursion operators. We will calculate them using the bilocal quantity $\varphi(x,\tilde{y},y,t)$ and proceed to $\tilde{y}=y$ at the very end.

Rewrite (2.5) in the form

$$\begin{aligned}\left[A, \frac{\partial \varphi(x,\tilde{y},y,t)}{\partial \tilde{y}}\right] = -\frac{\partial \varphi(x,\tilde{y},y,t)}{\partial x} - \left(\frac{\partial}{\partial \tilde{y}} + \frac{\partial}{\partial y}\right)\varphi(x,\tilde{y},y,t)A - \\ - P'(x,\tilde{y},t)\,\varphi(x,\tilde{y},y,t) + \varphi(x,\tilde{y},y,t)P(x,y,t)\end{aligned} \tag{3.1}$$

Acting on (3.1) by projection operation Δ_k and taking into account (2.2), we obtain

$$\begin{aligned}A\partial_{\tilde{y}}\varphi_{\Delta_k} - \delta_{k1}\partial_{\tilde{y}}\varphi_{\Delta_N}\cdot A = -\partial_x\varphi_{\Delta_k} - (\partial_{\tilde{y}}+\partial_y)\delta_{k1}\varphi_{\Delta_N}\cdot A - \\ - P'\varphi_{\Delta_k} + (\varphi_{\Delta_N}\cdot P)_{\Delta_k} - (1-\delta_{k1})\,\varphi_{\Delta_{k-1}}\cdot P_\infty \quad .\end{aligned} \tag{3.2}$$

Solving the recursion relations (3.2) with respect to φ_{Δ_k}, one gets

$$\varphi_{\Delta_k} = \mathcal{P}^{N-k}\varphi_{\Delta_N}(A^T+P_\infty^T)^{N-k} + \sum_{m=0}^{N-k-1}\mathcal{P}^m(\varphi_{\Delta_N}\cdot P)_{\Delta_{k+m+1}}(A^T+P_\infty^T)^{m+1} \tag{3.3}$$

$$(1 \leqslant k \leqslant N-1)$$

where operator $\mathcal{P}$ is

$$\mathcal{P} = -\partial_x - P'(x,\tilde{y},t) - A\partial_{\tilde{y}} \tag{3.4}$$

Substituting φ_{Δ_1} given by (3.3) into (3.2) for $k = 1$ we obtain the relation which contains only φ_{Δ_N} :

$$\sum_{\ell=0}^{N}\mathcal{P}^\ell\varphi_{\Delta_N}\circ V_\ell = -\partial_{\tilde{y}}\varphi_{\Delta_N} + (\partial_y+\partial_{\tilde{y}})\varphi_{\Delta_N} \tag{3.5}$$

where $V_N \equiv 1$ and $(\varphi\circ V_m)_{ik} \overset{def}{=} \varphi_{ik}V_m$.

In virtue of (2.2) the operator $\mathcal{P}^\ell$ is of the form

$$\mathcal{P}^\ell = \tilde{\mathcal{P}}^\ell - \sum_{k_1+k_2=\ell-1}\tilde{\mathcal{P}}^{k_1}A\tilde{\mathcal{P}}^{k_2}\partial_{\tilde{y}} \equiv \tilde{\mathcal{P}}^\ell - r_\ell\partial_{\tilde{y}} \tag{3.6}$$

where

$$\tilde{\mathcal{P}} \overset{def}{=} -\partial_x - P'(x,\tilde{y},t), \quad r_\ell \overset{def}{=} \sum_{k_1+k_2=\ell-1}\tilde{\mathcal{P}}^{k_1}A\tilde{\mathcal{P}}^{k_2} . \tag{3.7}$$

Let us introduce the N -component columns

$$\begin{aligned}V(x,y,t) \overset{def}{=} (V_0(x,y,t),\ldots,V_{N-1}(x,y,t))^T, \\ \chi(x,\tilde{y},y,t) \overset{def}{=} (\varphi_{1N}(x,\tilde{y},y,t),\ldots,\varphi_{NN}(x,\tilde{y},y,t))^T.\end{aligned} \tag{3.8}$$

Substituting (3.6) into (3.5) and transiting to the columns (3.8), we obtain

$$\mathcal{Y}\partial_{\tilde{y}}\chi(x,\tilde{y},y,t) = \mathcal{T}\chi(x,\tilde{y},y,t) \tag{3.9}$$

where

$$\mathcal{Y} \overset{def}{=} \sum_{\ell=0}^{N} r_\ell V_\ell - 1, \quad \mathcal{T} \overset{def}{=} \sum_{m=0}^{N}\tilde{\mathcal{P}}^m V_m - (\partial_y + \partial_{\tilde{y}}) . \tag{3.10}$$

From (3.10) it follows that the operator $\mathcal{Y}$ is a lowertriangular one: $\mathcal{Y}_{ik} = 0$, $k \geqslant i$ $(i,k=1,\ldots,N)$ and in particular $\mathcal{Y}_{i\,i-1} = -N\partial_x + V_{N-1}(x,y,t) - V'_{N-1}(x,\tilde{y},t)$. It is clear from (3.10) that the relation (3.9) allows us to express $\partial_{\tilde{y}}\chi(x,\tilde{y},y,t)$ through $\chi(x,\tilde{y},y,t)$. However since the operator $\mathcal{Y}$ is degenerate one then the first equation (3.9) give a constraint between the components of $\chi(x,\tilde{y},y,t)$.

Using (3.10) and the expressions

$$(\tilde{\mathcal{P}}^{k})_{1\ell}=C_{k}^{\ell-1}(-\partial_x)^{k+1-\ell},\ \ell=1,\dots,k+1,$$

$$(\tilde{\mathcal{P}}^{k})_{1\ell}=0,\ \ell=k+2,\dots,N,\ (k=1,\dots,N-1); \qquad (3.11)$$

$$(\tilde{\mathcal{P}}^{N})_{1\ell}=C_{N}^{\ell-1}(-\partial_x)^{N+1-\ell}-V'_{\ell-1}(x,\tilde{y},t),\ \ell=1,\dots,N,$$

we get this constraint in the form

$$\sum_{k=1}^{N}\ell_k\chi_k(x,\tilde{y},y,t)=0 \qquad (3.12)$$

where operators ℓ_k act as follows

$$\ell_k=(\partial_y+\partial_{\tilde{y}})\delta_{k1}-V_{k-1}(x,y,t)+V'_{k-1}(x,\tilde{y},t)-\sum_{n=1}^{N-k+1}C_{n+k-1}^{k-1}(-\partial_x)^{n}V_{n+k-1}(x,y,t). \qquad (3.13)$$

Solving (3.12) with respect to χ_N, we obtain the relation

$$\chi(x,\tilde{y},y,t)=M\chi_{\Delta}(x,\tilde{y},y,t) \qquad (3.14)$$

where

$$M_{ik}=\delta_{ik}-\delta_{iN}\ell_N^{-1}\ell_k,\ (i,k=1,\dots,N) \qquad (3.15)$$

and $\chi_{\Delta}\overset{def}{=}(\chi_1(x,\tilde{y},y,t),\dots,\chi_{N-1}(x,\tilde{y},y,t),0)^{T}$.

It is not difficult also to show that the following relation

$$[\partial_{\tilde{y}}^{n},M]=\sum_{p=0}^{n-1}M_{(n-1,p)}\partial_{\tilde{y}}^{p}+\Delta M \qquad (3.16)$$

holds, where

$$(M_{(n-1,p)})_{km}=-\delta_{kN}\ell_N^{-1}C_n^{p}V'_{m-1(n-p)}(x,\tilde{y},t),$$

$$(\Delta M)_{km}=\delta_{kN}\ell_N^{-1}\sum_{p=0}^{n-1}C_n^{p}V'_{N-1(n-p)}(x,\tilde{y},t)\partial_{\tilde{y}}^{p}\ell_N^{-1}\ell_m \qquad (3.17)$$

and $V'_{m(n-p)}\overset{def}{=}\partial_{\tilde{y}}^{n-p}V'_m(x,\tilde{y},t)$. Note that, by virtue of

(3.12), one has

$$\Delta M\chi(x,\tilde{y},y,t)=0. \qquad (3.18)$$

Further, acting on (3.9) by $\partial_{\tilde{y}}^{n}$, using (3.10), we find

$$\mathcal{G}\,\partial_{\tilde{y}}^{n+1}\chi(x,\tilde{y},y,t)=\sum_{m=0}^{n}\mathcal{F}_{(n,m)}\partial_{\tilde{y}}^{m}\chi(x,\tilde{y},y,t) \qquad (3.19)$$

where

$$\mathcal{F}_{(n,m)}=-(\partial_y+\partial_{\tilde{y}})\delta_{nm}+\sum_{\ell=0}^{N}C_n^{m}(\partial_{\tilde{y}}^{n-m}\tilde{\mathcal{P}}^{\ell})V_\ell(x,y,t)-$$

$$-(1-\delta_{m0})\sum_{\ell=0}^{N}C_n^{m-1}(\partial_{\tilde{y}}^{n-m+1}r_\ell)V_\ell(x,y,t). \qquad (3.20)$$

Then, using (3.7), it is easy to show that

$$\partial_{\tilde{y}}^{n-m+1}r_\ell=(1-\delta_{\ell0})\sum_{k_1+k_2=\ell-1}(\partial_{\tilde{y}}^{n-m+1}\tilde{\mathcal{P}}^{k_1})A\tilde{\mathcal{P}}^{k_2}. \qquad (3.21)$$

Introduce an operator $\tilde{\mathcal{G}}$ which satisfies to the equation $\tilde{\mathcal{G}}\mathcal{G}=E_N$ where $(E_N)_{ik}=\delta_{ik}-\delta_{iN}\delta_{kN}$, $(i,k=1,\dots,N)$.
Multipling (3.19) by this operator $\tilde{\mathcal{G}}$ and using (3.14), we finally obtain

$$\partial_{\tilde{y}}^{n+1}\chi_{\Delta}(x,\tilde{y},y,t)=\sum_{p=0}^{n}\sum_{m=p}^{n}\tilde{\mathcal{G}}\mathcal{F}_{(n,m)}C_m^{p}(\partial_{\tilde{y}}^{m-p}M)\partial_{\tilde{y}}^{p}\chi_{\Delta}(x,\tilde{y},y,t). \qquad (3.22)$$

The relation (3.22) contains only N-1 independent components $\chi_{\Delta}(x,\tilde{y},y,t)\overset{def}{=}(\chi_1(x,\tilde{y},y,t),\dots,\chi_{N-1}(x,\tilde{y},y,t),0)^{T}$ and can serve for the definition of the recursion operators in the space of N-1 independent variables.

Recursion operators can be defined in the space of all N components of $\chi(x,\tilde{y},y,t)$ too. For this purpose we multiple (3.19) from the left by $M\tilde{\mathcal{G}}$ and use (3.16) and (3.18). As a result we obtain

$$\partial_{\tilde{y}}^{n+1}\chi(x,\tilde{y},y,t)=\sum_{m=0}^{n}(M\tilde{\mathcal{G}}\mathcal{F}_{(n,m)}+M_{(n,m)})\partial_{\tilde{y}}^{m}\chi(x,\tilde{y},y,t). \qquad (3.23)$$

There is a simple relation between $\partial_{\tilde{y}}^{m}\chi(x,\tilde{y},y,t)$ and $\partial_{\tilde{y}}^{\ell}\chi_{\Delta}(x,\tilde{y},y,t)$. Indeed, acting on (3.14) by $\partial_{\tilde{y}}^{m}$, one gets

$$\partial_{\tilde{y}}^{m}\chi(x,\tilde{y},y,t)=\sum_{\ell=0}^{m}C_m^{\ell}(\partial_{\tilde{y}}^{m-\ell}M)\partial_{\tilde{y}}^{\ell}\chi_{\Delta}(x,\tilde{y},y,t)=\sum_{\ell=0}^{m}O_{(m,\ell)}\partial_{\tilde{y}}^{\ell}\chi_{\Delta}. \qquad (3.24)$$

Using (3.16), one can obtain the following convenient recursion relation for the calculation of $O_{(m,\ell)}$:

$$O_{(n+1,\ell)} = M\delta_{\ell n+1} + (1-\delta_{\ell n+1})\sum_{m=\ell}^{n} M_{(n,m)} O_{(m,\ell)} . \qquad (3.25)$$

The equations (3.22) and (3.23) allow to express all derivatives $\partial_{\tilde{y}}^{n}\chi(x,\tilde{y},y,t)$ through the quantity $\chi(x,\tilde{y},y,t)$. Therefore we can define the following operators $\hat{\Lambda}_n$ and $\hat{\Lambda}_{n(N)}$

$$\partial_{\tilde{y}}^{n}\chi(x,\tilde{y},y,t)\big|_{\tilde{y}=y} \stackrel{def}{=} \hat{\Lambda}_n\chi(x,y,t) , \qquad (3.26)$$

$$\partial_{\tilde{y}}^{n}\chi_{\Delta}(x,\tilde{y},y,t)\big|_{\tilde{y}=y} \stackrel{def}{=} \hat{\Lambda}_{n(N)}\chi_{\Delta}(x,y,t) \qquad (3.27)$$

where $\chi(x,y,t) \stackrel{def}{=} \chi(x,\tilde{y},y,t)\big|_{\tilde{y}=y}$ and $\hat{\Lambda}_0 \equiv 1 \equiv \hat{\Lambda}_{0(N)}$. From (3.22) and (3.23) follow the recursion relations for the calculation of the operators $\hat{\Lambda}_n$ and $\hat{\Lambda}_{n(N)}$:

$$\hat{\Lambda}_{n+1} = \sum_{m=0}^{n}\mathcal{D}_{(n,m)}\hat{\Lambda}_m + Q_{(n+1)}\otimes\ell , \qquad (3.28)$$

$$\hat{\Lambda}_{n+1(N)} = \sum_{m=0}^{n}\mathcal{D}_{(n,m)(N)}\hat{\Lambda}_{m(N)} \qquad (3.29)$$

where

$$\mathcal{D}_{(n,m)} \stackrel{def}{=} (M\mathcal{G}\mathcal{F}_{(n,m)} + M_{(n,m)})\big|_{\tilde{y}=y} ,$$
$$\mathcal{D}_{(n,m)(N)} \stackrel{def}{=} \sum_{\ell=m}^{n}\mathcal{G}\mathcal{F}_{(n,\ell)}O_{(\ell,m)}\big|_{\tilde{y}=y} \qquad (3.30)$$

and $\ell = (\ell_1,\dots,\ell_N)\big|_{\tilde{y}=y}$, $Q_{(n+1)} = (Q_{(n+1)1},\dots,Q_{(n+1)N})^T$ where $Q_{(n+1)k}$ are arbitrary operators.

The operators $\hat{\Lambda}_n$ and $\hat{\Lambda}_{n(N)}$ are just the recursion operators we are interesting.

The terms $Q_{(n+1)}\otimes\ell$ in (3.28) is due to the existence of the constraint (3.12). Indeed in virtue of (3.23), the quantity $d_{n+1} = \hat{\Lambda}_{n+1} - \sum_{m=0}^{n}\mathcal{D}_{(n,m)}\hat{\Lambda}_m$ should satisfy to the equation $d_{n+1}\chi(x,y,t)=0$. Since χ has N-1 independent components then the rank of operator d_{n+1} is equal to 1. Taking into account the constraint $\sum_{k=1}^{N}\ell_k\chi_k = 0$, we obtain $d_{(n+1)ik} = Q_{(n+1)i}\,\ell_k$ where $Q_{(n+1)i}$ are arbitrary operators.

It is convenient also to introduce "standard" recursion

operators $\hat{\Lambda}_{(S)n}$ which are defined by the recursion relations

$$\hat{\Lambda}_{(S)n+1} = \sum_{m=0}^{n}\mathcal{D}_{(n,m)}\hat{\Lambda}_{(S)m} . \qquad (3.31)$$

The relation between $\hat{\Lambda}_n$ and $\hat{\Lambda}_{(S)n}$ is the following

$$\hat{\Lambda}_n = \hat{\Lambda}_{(S)n} + Q_{(n)}\otimes\ell \qquad (3.32)$$

where $Q_{(n)} \stackrel{def}{=} (Q_{(n)1},\dots,Q_{(n)N})^T$ and $Q_{(n)1},\dots,Q_{(n)N}$ are arbitrary operators.

Thus if one defines the recursion operators on the N-1-dimensional space of independent quantities $\chi_{\Delta} = (\chi_1,\dots,\chi_{N-1},0)^T$ the recursion operators $\hat{\Lambda}_{n(N)}$ are defined uniquely by formulas (3.29), (3.30). If one defines recursion operators on the whole N-dimensional space of all components $\chi_1,\dots,\chi_N$ then there exist a big uncertainty in the form of recursion operators due to the terms $Q_{(n)}\otimes\ell$. Such a situation with the different possible definitions of recursion operators and related uncertainty is a typical one for AKNS method. In the onedimensional case it was demonstrated for Gelfand-Dikij spectral problem [16] and for the linear arbitrary order matrix spectral problem [20].

In analogous way one can obtain the recursion relations for calculation of the recursion operators $\check{\Lambda}_n$ and $\check{\Lambda}_{n(N)}$ defined by the formulas

$$\check{\Lambda}_n\chi(x,y,t) \stackrel{def}{=} \partial_y^n\chi(x,\tilde{y},y,t)\big|_{\tilde{y}=y} , \quad \check{\Lambda}_{n(N)}\chi_{\Delta}(x,y,t) \stackrel{def}{=} \partial_y^n\chi_{\Delta}(x,\tilde{y},y,t)\big|_{\tilde{y}=y} . \qquad (3.33)$$

However it is more convenient to calculate the operators $\check{\Lambda}_n$ and $\check{\Lambda}_{n(N)}$ using the obvious formulas

$$\check{\Lambda}_n = \sum_{k=0}^{n} C_n^k(-1)^k\partial_y^{n-k}\hat{\Lambda}_k , \quad \check{\Lambda}_{n(N)} = \sum_{k=0}^{n}(-1)^k C_n^k\partial_y^{n-k}\hat{\Lambda}_{k(N)} . \qquad (3.34)$$

For further constructions we will need the operators $\hat{\Lambda}_n^+$, $\hat{\Lambda}_{n(N)}^+$, $\check{\Lambda}_n^+$ and $\check{\Lambda}_{n(N)}^+$ adjoint to the operators $\hat{\Lambda}_n$, $\hat{\Lambda}_{n(N)}$, $\check{\Lambda}_n$, $\check{\Lambda}_{n(N)}$ with respect to the bilinear form $\langle\langle\chi'\chi\rangle\rangle = \int_{-\infty}^{+\infty} dx\,dy\sum_{i=1}^{N}\chi_i'(x,y,t)\chi_i(x,y,t)$. The recurrent relations for the calculations of $\hat{\Lambda}_{(S)n}^+$ and $\hat{\Lambda}_{n(N)}^+$ are the following

$$\hat{\Lambda}^{+}_{(S)n+1}=\sum_{m=0}^{n}\hat{\Lambda}^{+}_{(S)(m,n)}\mathcal{D}^{+}_{(n,m)},\ \hat{\Lambda}^{+}_{n+1(N)}=\sum_{m=0}^{n}\hat{\Lambda}^{+}_{m(N)}\mathcal{D}^{+}_{(n,m)(N)} \tag{3.35}$$

where

$$\mathcal{D}^{+}_{(n,m)}=\left(\mathcal{T}^{+}_{(n,m)}\tilde{\mathcal{Y}}^{+}M^{+}+M^{+}_{(n,m)}\right)\Big|_{\tilde{y}=y},$$

$$\mathcal{D}^{+}_{(n,m)(N)}=\sum_{\ell=m}^{n}O^{+}_{(\ell,m)}\mathcal{F}^{+}_{(n,\ell)}\tilde{\mathcal{Y}}^{+}\Big|_{\tilde{y}=y}. \tag{3.36}$$

Then

$$\hat{\Lambda}^{+}_{n}=\hat{\Lambda}^{+}_{(S)n}+\ell^{+}\otimes Q^{+}_{(n)} \tag{3.37}$$

and

$$\check{\Lambda}^{+}_{n}=\sum_{k=0}^{n}(-1)^{n}\hat{\Lambda}^{+}_{k}\partial_y^{n-k},\quad \check{\Lambda}^{+}_{n(N)}=\sum_{k=0}^{n}(-1)^{n}\hat{\Lambda}^{+}_{k(N)}\partial_y^{n-k}. \tag{3.38}$$

Operators $O^{+}_{(n,m)}$ are calculated by recurrent relations

$$O^{+}_{(n+1,m)}=\delta_{m\,n+1}M^{+}+(1-\delta_{m\,n+1})\sum_{k=m}^{n}O^{+}_{(k,m)}M^{+}_{(n,k)}. \tag{3.39}$$

Using (3.35)-(3.37) and (3.39) one can also show by induction that

$$M^{+}\hat{\Lambda}^{+}_{k}=\sum_{m=0}^{k}\hat{\Lambda}^{+}_{m(N)}O^{+}_{(k,m)}. \tag{3.40}$$

IV. General form of Backlund transformations and integrable equations

In the previous section it was shown that all matrix elements of $\varphi(x,\tilde{y},y,t)$ can be expressed through φ_{Δ_N}. Analogously we can express $\partial_{\tilde{y}}^{n}\varphi_{\Delta_{N-k}}(x,\tilde{y},y,t)$ in the first term in (2.13) through $\chi(x,\tilde{y},y,t)$.

Starting equation for this calculation is equation (2.5) which we rewrite in the form

$$\left[A,\frac{\partial\varphi(x,\tilde{y},y,t)}{\partial y}\right]=\frac{\partial\varphi(x,\tilde{y},y,t)}{\partial x}+A\left(\frac{\partial}{\partial\tilde{y}}+\frac{\partial}{\partial y}\right)\varphi(x,\tilde{y},y,t)+ \tag{4.1}$$

$$+P'(x,\tilde{y},t)\varphi(x,\tilde{y},y,t)-\varphi(x,\tilde{y},y,t)P(x,y,t).$$

Applying the operator Δ_k to (4.1) and performing the calculations analogous to those given at the beginning of section 3, we obtain from (4.1) the following expression for φ_{Δ_k}:

$$\varphi_{\Delta_k}=\mathcal{T}^{N-k}\varphi_{\Delta_N}(A^{T}+P^{T}_{\infty})^{N-k}+\sum_{m=0}^{N-k+1}\mathcal{T}^{m}(\varphi_{\Delta_N}\cdot P)_{\Delta_{k+m+1}}(A^{T}+P^{T}_{\infty})^{m+1} \tag{4.2}$$

where

$$\mathcal{T}=-\partial_x-P'(x,\tilde{y},t)-A(\partial_{\tilde{y}}+\partial_y)+A\partial_y. \tag{4.3}$$

It is not difficult to show that

$$\mathcal{T}^{\ell}=\tilde{\mathcal{T}}^{\ell}+t_{\ell}\partial_y \tag{4.4}$$

where

$$\tilde{\mathcal{T}}\overset{def}{=}-\partial_x-P'(x,\tilde{y},t)-A(\partial_{\tilde{y}}+\partial_y),\quad t_{\ell}\overset{def}{=}\sum_{k_1+k_2=\ell-1}\tilde{\mathcal{T}}^{k_1}A\tilde{\mathcal{T}}^{k_2}.$$

Substitute $\varphi_{\Delta_{N-k}}$ from (4.2) into (2.13) and integrate over $\tilde{y}$. Using (4.4), (3.26), (3.27), (3.33), (3.35), passing to N-component columns V and χ (see (3.8)) and to the adjoint operators we obtain from (2.13) the following equation

$$\left\langle\!\left\langle \chi\left\{\sum_{k=0}^{N-1}\sum_{n=0}^{\infty}b_{kn}(t)\left(\mathcal{K}^{+}_{(k,n)}V'-\mathcal{M}^{+}_{(k,n)}V\right)\right\rangle\!\right\rangle=0, \tag{4.5}$$

where

$$\mathcal{K}^{+}_{(k,n)}=\sum_{m=0}^{k}\sum_{p=0}^{n+1}\sum_{q=0}^{p}C^{p}_{n+1}C^{q}_{p}(-1)^{p}\hat{\Lambda}^{+}_{q}\partial_y^{p-q}V_{N-m(n+1-p)}t^{+}_{k-m}+$$

$$+\sum_{m=0}^{k}\sum_{p=0}^{n}\sum_{q=0}^{p}C^{p}_{n}C^{q}_{p}(-1)^{p}\hat{\Lambda}^{+}_{q}\partial_y^{p-q}V_{N-m(n-p)}(\tilde{\mathcal{T}}^{+})^{k-m}, \tag{4.6}$$

$$\mathcal{M}^{+}_{(k,n)}=(-1)^{n+1}\hat{\Lambda}^{+}_{n+1}(P_{\infty})^{N-k}+(-1)^{k}\hat{\Lambda}^{+}_{n}(P^{T}_{\infty})^{k}.$$

Note that in all the quantities contained in (4.5) and (4.6) and below $\tilde{y}=y$. The designations of the quantities which were introduced at $\tilde{y}\neq y$ will not be changed.

The components χ_k in (4.5) are not independent ones due to the constraint (3.12). Therefore from (4.5) it follows

$$\sum_{k=0}^{N-1}\sum_{n=0}^{\infty} b_{kn}(t)\left(\mathcal{K}^{+}_{(k,n)}V' - \mathcal{M}^{+}_{(k,n)}V\right) - e^{+}\phi = 0 \qquad (4.7)$$

where $e^{+} = (e_1^{+}, \ldots, e_N^{+})^{T}$ and $\phi(x,y,t)$ is arbitrary scalar function.

Indeed from (4.5) follows that $\sum_{k=0}^{N-1}\sum_{n=0}^{\infty} b_{kn}(t)\left(\mathcal{K}^{+}_{(k,n)}V' - \mathcal{M}^{+}_{(k,n)}V\right) = Z^{+}$ where Z^{+} is a column such that $Z\chi = 0$. The general form of such Z is $Z_k = \phi(x,y,t)e_k$ where $\phi(x,y,t)$ is arbitrary scalar function. As a result $Z_k^{+} = e_k^{+}\phi$.

Substituting the relations (3.37) into $\mathcal{K}^{+}_{(k,n)}$, $\mathcal{M}^{+}_{(k,n)}$, one can show that (4.7) is equivalent to equation

$$\sum_{k=0}^{N-1}\sum_{n=0}^{\infty} b_{kn}(t)\left(\mathcal{K}^{+}_{(S)(k,n)}V' - \mathcal{M}^{+}_{(S)(k,n)}V\right) - e^{+}\phi = 0 \qquad (4.8)$$

where ϕ is an arbitrary scalar function and $\mathcal{K}^{+}_{(S)(k,n)}$, $\mathcal{M}^{+}_{(S)(k,n)}$ are given by formulas

$$\mathcal{K}^{+}_{(S)(k,n)} = \mathcal{K}^{+}_{(k,n)}\Big|_{\hat{\Lambda}^{+}_{m} \to \hat{\Lambda}^{+}_{(S)m}}, \quad \mathcal{M}^{+}_{(S)(k,n)} = \mathcal{M}^{+}_{(k,n)}\Big|_{\hat{\Lambda}^{+}_{m} \to \hat{\Lambda}^{+}_{(S)m}}. \qquad (4.9)$$

Operators $\hat{\Lambda}^{+}_{(S)m}$, $\mathcal{K}^{+}_{(S)(k,n)}$, $\mathcal{M}^{+}_{(S)(k,n)}$ are defined uniquely and all uncertainty in (4.8) is contained in the term $e^{+}\phi$.

Multiplying (4.8) by M^{+} and taking into account (3.40) and equality $M^{+}e^{+} = 0$, we obtain from (4.8) the equation

$$\sum_{k=0}^{N-1}\sum_{n=0}^{\infty} b_{kn}(t)\left(\mathcal{K}^{+}_{(k,n)(N)}V' - \mathcal{M}^{+}_{(k,n)(N)}V\right) = 0 \qquad (4.10)$$

where operators $\mathcal{K}^{+}_{(k,n)(N)}$ and $\mathcal{M}^{+}_{(k,n)(N)}$ act as follows

$$\mathcal{K}^{+}_{(k,n)(N)} = \sum_{m=0}^{k}\sum_{p=0}^{n}\sum_{q=0}^{p}\sum_{r=0}^{q} C_n^p C_p^q (-1)^p \hat{\Lambda}^{+}_{r(N)} O^{+}_{(q,r)} \partial_y^{p-q} V_{N-m(n-p)} (\widetilde{\mathcal{T}}^{+})^{k-m} +$$
$$+ \sum_{m=0}^{k}\sum_{p=0}^{n+1}\sum_{q=0}^{p}\sum_{r=0}^{q} C_{n+1}^p C_p^q (-1)^p \hat{\Lambda}^{+}_{r(N)} O^{+}_{(q,r)} \partial_y^{p-q} V_{N-m(n+1-p)}\, t^{+}_{k-m}, \qquad (4.11)$$
$$\mathcal{M}^{+}_{(k,n)(N)} = (-1)^{n+1}\sum_{m=0}^{n+1} \hat{\Lambda}^{+}_{m(N)} O^{+}_{(n+1,m)} (P_\infty)^{N-k} + (-1)^n \sum_{m=0}^{n} \hat{\Lambda}^{+}_{m(N)} O^{+}_{(n,m)} (P_\infty^{T})^{k}.$$

The uncertainty which is contained in (4.8) disappear after the transition to (4.10). The relation (4.10) due to the special form of the operators $\hat{\Lambda}^{+}_{n(N)}$ (see (3.25),(3.36)) contains N-1 nontrivial equations. In contrast, the relation (4.8) contains N nontrivial equations.

Let us consider in more details the transformations (4.8) (or (4.10)). The corresponding transformation of the scattering matrix is given by (2.12). One can show that the transformations (2.12), (4.8) (or (4.10)) form an abelian infinite-dimensional group. We will refer this group as Backlund-Calogero (BC) group (for motivation see [16]). On the manifold of scattering matrices $\{S(\bar{\lambda},\lambda,t)\}$ action of BC group is given by formula (2.12). Formula (4.8) (or (4.10)) determine action of BC group on the manifold of potentials $\{V(x,y,t)\}$.

BC group contains the transformations of various types. Let us consider an infinitesimal displacement in time t: $t \to t' = t + \varepsilon$, $\varepsilon \to 0$. In this case

$$V'(x,y,t) = V(x,y,t') = V(x,y,t) + \varepsilon \frac{\partial V(x,y,t)}{\partial t}, \qquad (4.12)$$

$$B_k(\lambda^N,t) = \delta_{k0} - \varepsilon\Omega_k(\lambda^N,t) = \delta_{k0} - \varepsilon\sum_{n=0}^{\infty}\omega_{kn}(t)\lambda^{Nn}.$$

$$\phi(x,y,t) = \varepsilon\varphi(x,y,t). \qquad (4.13)$$

Substituting (4.12), (4.13) into (4.8) and keeping the terms of the first order on ε, we obtain

$$\frac{\partial V(x,y,t)}{\partial t} - \sum_{k=1}^{N-1}\sum_{n=0}^{\infty}\omega_{kn}(t)\mathcal{L}^{+}_{(S)(k,n)}V - e^{+}\varphi = 0 \qquad (4.14)$$

where

$$\mathcal{L}^{+}_{(S)(k,n)} = \left(\mathcal{K}^{+}_{(S)(k,n)} - \mathcal{M}^{+}_{(S)(k,n)}\right)\Big|_{V'=V}. \qquad (4.15)$$

System of N equations (4.14) just represents the general form of the nonlinear evolution equations integrable by the spectral problem (1.1) with the help of inverse scattering transform method. Transformations (4.8) are general Backlund transformations for the integrable equations (4.14).

Substituting (4.12) into (4.10) and using (4.11) one obtains

$$M^{+}\frac{\partial V(x,y,t)}{\partial t} - \sum_{k=1}^{N-1}\sum_{n=0}^{\infty}\omega_{kn}(t)\mathcal{L}^{+}_{(k,n)(N)}V = 0 \qquad (4.16)$$

where

$$\mathcal{L}^+_{(k,n)(N)} = (\mathcal{K}^+_{(k,n)(N)} - \mathcal{M}^+_{(k,n)(N)})\big|_{V'=V}, \quad M^+ = M^+(V,V')\big|_{V'=V}. \quad (4.17)$$

In the onedimensional limit $\partial V_k/\partial y = 0$, $(k=0,1,\dots,N-1)$ all the transformations and equations (4.8), (4.10), (4.14), (4.16) convert into the transformations and equations constructed earlier in [16].

V. Gauge invariance and manifestly gauge invariant formulation of integrable equations and Backlund transformations

Spectral problem (1.1) is invariant under the following gauge transformations

$$\psi(x,y,t) \to \tilde{\psi}(x,y,t) = g(x,y,t)\psi(x,y,t),$$

$$V_k(x,y,t) \to \tilde{V}_k(x,y,t) = g(x,y,t)\Big\{\sum_{n=0}^{N-k} C^k_{n+k} V_{n+k}(x,y,t)\partial_x^n\Big(\frac{1}{g(x,y,t)}\Big) + \delta_{k0}\partial_y\Big(\frac{1}{g(x,y,t)}\Big)\Big\} \quad (5.1)$$

where $g(x,y,t)$ is an arbitrary differentiable function such that $g(x,y,t)\xrightarrow[\sqrt{x^2+y^2}\to\infty]{} 1$. Transformations (5.1) form an infinitedimensional gauge group for problem (1.1).

There exist N-1 independent gauge invariants $W_k(\tilde{V}) = W_k(V)$, $k=0,1,\dots,N-2$. They are [21]

$$W_k = V_k - \frac{1}{N}\sum_{n=1}^{N-k} C^k_{n+k} V_{n+k}\Big(\partial_x - \frac{1}{N}V_{N-1}\Big)^{n-1} V_{N-1} - \frac{1}{N}\delta_{k0}\partial_x^{-1}\partial_y V_{N-1}. \quad (5.2)$$

Gauge invariance of (1.1) permit to impose an additional constraint on the potentials $V_0,\dots,V_{N-1}$ (gauge condition). For example, $V_{N-1}=0$ or $\sum_{k=0}^{N-1}\alpha_k V_k = 0$ where α_k are arbitrary constants.

Transformation law of the quantity $\Phi^{in}_{ke}(x,\tilde{y},y,t)$ under the gauge group is very simple

$$\Phi^{in}(x,\tilde{y},y,t) \xrightarrow{(g_1,g_2)} \tilde{\Phi}^{in}(x,\tilde{y},y,t) = G(g_2(x,\tilde{y},t))\,\Phi^{in}(x,\tilde{y},y,t)\,G^{-1}(g_1(x,y,t)) \quad (5.3)$$

where $G_{ik} = C^{k-1}_{i-1}\partial_x^{i-k} g(x,y,t)$ for $i \geq k$ and $G_{ik}=0$ for $i<k$ $(i,k=1,\dots,N)$. In formula (5.3) and below, the quantities which correspond to different potentials and potentials $V(x,y,t)$, $V'(x,\tilde{y},t)$ themselves are transformed under the gauge transformations with different gauge functions $g_1(x,y,t)$ and $g_2(x,\tilde{y},t)$.

Using (5.3), (3.14) and definitions (3.26), (3.27), we obtain the transformation laws of the recursion operators $\hat{\Lambda}_{(S)n}$, $\hat{\Lambda}_{n(N)}$ and operators ℓ_n:

$$\hat{\Lambda}_{(S)n} \xrightarrow{(g_1,g_2)} \tilde{\hat{\Lambda}}_{(S)n} = \pi(g_1,g_2)\Big(\sum_{k=0}^{n} G(g_2)^{-1} C^k_n \big(\partial_y^{n-k} G(g_2)\big)\hat{\Lambda}_{(S)k}\Big)\pi^{-1}(g_1,g_2) + Q_{(n)}\otimes\ell,$$

$$\hat{\Lambda}_{n(N)} \xrightarrow{(g_1,g_2)} \tilde{\hat{\Lambda}}_{n(N)} = \pi(g_1,g_2)\Big(\sum_{k=0}^{n} G(g_2)^{-1} C^k_n \big(\partial_y^{n-k} G(g_2)\big)\hat{\Lambda}_{k(N)}\Big)\pi^{-1}(g_1,g_2). \quad (5.4)$$

$$\sum_{n=1}^{N} \ell_n \pi_{nm}(g_1,g_2) = \frac{g_2}{g_1}\ell_m, \quad (m=1,\dots,N) \quad (5.5)$$

where $\pi(g_1,g_2) = G(g_2(x,y,t))/g_1(x,y,t)$. Operator $Q_{(n)} = (Q_{(n)1},\dots,Q_{(n)N})^T$ in (5.4) is determined from the concrete form of the operators $\hat{\Lambda}_{(S)n}$.

Using the transformation laws of P and P' with different gauge functions $g_1(x,y,t)$ and $g_2(x,\tilde{y},t)$ and (5.3) one can prove that the relation (2.9) is gauge invariant one. From gauge invariance of (2.9) and (5.5) we obtain the following transformation law of the nonlinear transformations (4.8):

$$\sum_{k=0}^{N-1}\sum_{n=0}^{\infty} b_{kn}(t)\big(\tilde{\mathcal{K}}^+_{(S)(k,n)}\tilde{V}' - \tilde{\mathcal{M}}^+_{(S)(k,n)}\tilde{V}\big) - \ell^+\tilde{\phi} =$$
$$= (\pi^+(g_1,g_2))^{-1}\Big\{\sum_{k=0}^{N-1}\sum_{n=0}^{\infty} b_{kn}(t)\big(\mathcal{K}^+_{(S)(k,n)}V' - \mathcal{M}^+_{(S)(k,n)}V\big) - \ell^+\phi\Big\}. \quad (5.6)$$

In (5.6) $\tilde{\phi} = \frac{g_1}{g_2}\phi + \Delta\phi$ where term $\frac{g_1}{g_2}\phi$ is due to the gauge transformation of $\ell^+\phi$, $\Delta\phi$ is related with the transformation of the quantities $\mathcal{K}^+_{(S)(k,n)}V' - \mathcal{M}^+_{(S)(k,n)}V$ and depends on the concrete form of transformation (4.8).

Analogously to the onedimensional case [16] one can show that the transformations (4.8) of BC group contain the manifestly gauge invariant part

$$\sum_{k=0}^{N-1}\sum_{n=0}^{\infty} b_{kn}(t)\big(\mathcal{K}^+_{(k,n)(N)}W' - \mathcal{M}^+_{(k,n)(N)}W\big) = 0. \quad (5.7)$$

where all operators in (5.7) are given by the formulas (4.11) with the substitution $V_k \to W_k\,(k=0,\dots,N-2)$, $V_{N-1} \to 0$.

Note that the BC group which was constructed in section 4 contains a gauge group (5.1) as the subgroup. Indeed, let us consider the transformation (4.8) with $b_{kn}=\delta_{k0}\delta_{n0}$. In this case from (4.8) we obtain $V' = V + \ell^+\phi$, i.e.

$$V'_{k-1} = V_{k-1} + \delta_{k1}\frac{1}{1-\phi}\partial_y(1-\phi) + \frac{1}{1-\phi}\sum_{n=1}^{N-k+1} C^{k-1}_{n+k-1} V_{n+k-1}\partial_x^n(1-\phi)$$

that is coincide with gauge transformation (5.1) with gauge function $g=(1-\phi)^{-1}$.

In conclusion we shortly consider the gauge properties of the integrable equations (4.14). In this case $V=V'$ and therefore one must put $g_1=g_2$ in the transformation laws given above. Transformation law of the integrable equations (4.14) under gauge transformations (5.1) is the following

$$\frac{\partial\tilde V}{\partial t} - \sum_{k=1}^{N-1}\sum_{n=0}^{\infty}\omega_{kn}(t)\tilde{\mathcal{L}}^+_{(S)(k,n)}\tilde V - \tilde\ell^+\tilde\varphi = $$
$$= \tau^{-1}(g)\Big(\frac{\partial V}{\partial t} - \sum_{k=1}^{N-1}\sum_{n=0}^{\infty}\omega_{kn}(t)\mathcal{L}^+_{(S)(k,n)}V - \ell^+\varphi\Big) \qquad (5.8)$$

where $\tau(g) = g(G^T)^{-1}$ and $\tilde\varphi = \varphi + \Delta\varphi$ where $\Delta\varphi$ is due to the gauge transformation of $\mathcal{L}^+_{(S)(k,n)}V$ and depends on the concrete form of equation (4.14).

Manifestly gauge-invariant form of the integrable equations (4.16) is

$$\frac{\partial W}{\partial t} - \sum_{k=1}^{N-1}\sum_{n=0}^{\infty}\omega_{kn}(t)\mathcal{L}^+_{(k,n)(W)}W = 0 . \qquad (5.9)$$

VI. Examples: N = 2. Integrable equations and Backlund transformations

From (3.13) and (3.15) we have

$$\ell_1^+ = -\partial_y - V_1\partial_x - \partial_x^2 - V_0 + V_0' , \quad \ell_2^+ = -2\partial_x - V_1 + V_1' , \qquad (6.1)$$

$$(\ell_2^+)^{-1}\cdot = -\frac{1}{2}\exp\Big(\frac{1}{2}\int_x^{+\infty}(V_1-V_1')\Big)\int_{-\infty}^{x}dx'\exp\Big(-\frac{1}{2}\int_{x'}^{+\infty}(V_1-V_1')\Big)\cdot .$$

Gauge invariant $W_0 = V_0 - \frac{1}{2}\partial_x V_1 - \frac{1}{4}V_1^2 - \frac{1}{2}\partial_x^{-1}\partial_y V_1$.
Standard recursion operator $\hat\Lambda^+_{(S)1}$ is

$$\hat\Lambda^+_{(S)1} = \begin{pmatrix} -V_0' + (\partial_x V_0')\ell_2^{+-1}, & (-(\partial_x V_0') + V_0'\ell_2^+)\ell_2^{+-1}\ell_1^+\ell_2^{+-1} - (\partial_y V_0')\ell_2^{+-1} \\ -V_1' + (\ell_1^+ + (\partial_x V_1'))\ell_2^{+-1}, & (-\ell_1^+ - (\partial_x V_1') + V_1'\ell_2^+)\ell_2^{+-1}\ell_1^+\ell_2^{+-1} - (\partial_y V_1')\ell_2^{+-1} \end{pmatrix} . \qquad (6.2)$$

Backlund transformations (4.8), with $b_{00} = const$, $b_{10} = const$ and the rest b_{kn} equal to zero, have the form

$$b_{00}(V_0'-V_0) + b_{10}(\partial_x V_0')\exp\Big(-\frac{1}{2}\int_{-\infty}^{x}(V_1-V_1')\Big) + \partial_x^2\phi + V_1\partial_x\phi + (V_0-V_0')\phi + \partial_y\phi = 0,$$
$$b_{00}(V_1'-V_1) + b_{10}\Big[(\partial_x(V_1+V_1')) + \frac{1}{4}(V_1^2-V_1'^2) + V_0' - V_0 + \qquad (6.3)$$
$$+\frac{1}{2}\int_{-\infty}^{x}((\partial_y V_1)-(\partial_y V_1'))\Big]\exp\Big(-\frac{1}{2}\int_{-\infty}^{x}(V_1-V_1')\Big) + 2\partial_x\phi + (V_1-V_1')\phi = 0 .$$

Let us exclude function ϕ from (6.3). From second equation (6.3) we have $\phi(x,y,t) = b_{00} - \Big[b_{00} + \frac{1}{4}b_{10}(V_1'+V_1) +$
$$+\frac{1}{8}b_{10}\int_{-\infty}^{x}(V_1^2-V_1'^2) + \frac{b_{10}}{4}\int_{-\infty}^{x}dx'\int_{-\infty}^{x'}((\partial_y V_1)-(\partial_y V_1')) - \frac{b_{10}}{2}\int_{-\infty}^{x}(V_0-V_0')\Big]\times$$
$$\times\exp\Big(-\frac{1}{2}\int_{-\infty}^{x}(V_1-V_1')\Big) .$$

Substituting this expression for ϕ into the first equation (6.3) we obtain after calculations the gauge invariant part of Backlund transformations (6.3):

$$b_{00}(W_0'-W_0) + \frac{1}{2}b_{10}\partial_x(W_0'+W_0) - \frac{1}{2}b_{10}\int_{-\infty}^{x}(\partial_y(W_0'-W_0)) +$$
$$+\frac{b_{10}}{2}(W_0'-W_0)\int_{-\infty}^{x}(W_0'-W_0) = 0 \qquad (6.4)$$

where $W_0 = V_0 - \frac{1}{2}\partial_x V_1 - \frac{1}{4}V_1^2 - \frac{1}{2}\partial_x^{-1}\partial_y V_1$, $W_0' = V_0' - \frac{1}{2}\partial_x V_1' - \frac{1}{4}V_1'^2 - \frac{1}{2}\partial_x^{-1}\partial_y V_1'$. Relation (6.4) coincides with corresponding given by (5.6).

Present now some examples of integrable equations. Consider a system of equations (4.14) with nonzero ω_{10}, ω_{11} and $\omega_{kn}=0$, $k=2,3,\dots$. Calculating $\mathcal{L}^+_{(S)(1,0)}$ and $\mathcal{L}^+_{(S)(1,1)}$ by formulas (4.15), (4.9), (4.6), (4.17), we obtain the following system of equations

$$\frac{\partial V_0}{\partial t}=\omega_{10}(t)\partial_x V_0+\omega_{11}(t)\Big[\frac{3}{2}V_0\partial_x V_0-\frac{1}{2}\partial_y(V_0V_1)-\partial_x\partial_y V_0-$$

$$-\frac{1}{2}V_0\partial_x^2V_1-\frac{1}{2}V_0V_1\partial_xV_1-\frac{1}{4}(\partial_xV_0)(\partial_xV_1)-\frac{V_1^2}{8}(\partial_xV_0)-$$

$$-\frac{3}{4}(\partial_xV_0)\partial_x^{-1}\partial_yV_1\Big]-\big[\partial_y\varphi+V_1\partial_x\varphi+\partial_x^2\varphi\big], \tag{6.5}$$

$$\frac{\partial V_1}{\partial t}=\omega_{10}(t)\partial_xV_1+\omega_{11}(t)\Big[-\frac{1}{2}\partial_x^2V_0-\frac{3}{2}\partial_yV_0+\frac{1}{4}\partial_x^3V_1+$$

$$+\frac{1}{2}\partial_x(V_0V_1)-\frac{3}{8}V_1^2\partial_xV_1-\frac{3}{4}(\partial_xV_1)\partial_x^{-1}\partial_yV_1+\frac{3}{4}\partial_x^{-1}\partial_y^2V_1\Big]-2\partial_x\varphi.$$

Choosing in (6.5) $V_1=0$ from the second equation we obtain $\varphi=\omega_{11}(t)\big[-\frac{1}{4}(\partial_xV_0)-\frac{3}{4}\int_{-\infty}^{x}(\partial_yV_0)\big]$. Substituting this expression for φ into the first equation (6.5), one gets the Kadomtsev-Petviashvili equation [1]:

$$\frac{\partial V_0}{\partial t}=\omega_{10}(t)\partial_xV_0+\omega_{11}(t)\Big[\frac{1}{4}\partial_x^3V_0+\frac{3}{2}V_0\partial_xV_0+\frac{3}{4}\partial_x^{-1}\partial_y^2V_0\Big]. \tag{6.6}$$

Choosing in (6.5) $V_0=0$ from the first equation (6.5) we have $\partial_x^2\varphi+V_1\partial_x\varphi+\partial_y\varphi=0$. If one takes the solution $\varphi=const$ of this equation and substitute it into (6.5), one obtains modified Kadomtsev-Petviashvili equation [21]:

$$\frac{\partial V_1}{\partial t}=\omega_{10}(t)\partial_xV_1+\omega_{11}(t)\Big[\frac{1}{4}\partial_x^3V_1-\frac{3}{8}V_1^2\partial_xV_1-$$

$$-\frac{3}{4}(\partial_xV_1)\partial_x^{-1}\partial_yV_1+\frac{3}{4}\partial_x^{-1}\partial_y^2V_1\Big]. \tag{6.7}$$

Let us exclude function φ from (6.5) without fixing a gauge. It is not difficult to show that we obtain the gauge invariant part of the system (6.5):

$$\frac{\partial W_0}{\partial t}=\omega_{10}(t)\partial_xW_0+\omega_{11}(t)\Big[\frac{1}{4}\partial_x^3W_0+\frac{3}{2}W_0\partial_xW_0+\frac{3}{4}\partial_x^{-1}\partial_y^2W_0\Big] \tag{6.8}$$

that is the manifestly gauge invariant form of Kadomtsev-Petviashvili equation.

Consider now the general linear gauge $\alpha_0V_0+\alpha_1V_1=0$ where α_0 and α_1 are constants. Introduce function $u(x,y,t)$

such that $V_0=\beta_0u,\ V_1=\beta_1u,\ (\alpha_0\beta_0+\alpha_1\beta_1=0)$. By virtue of the equality $W_0(\tilde V_0,\tilde V_1)=W_0(V_0,V_1)$ we obtain the two dimensional Gardner transformation

$$\tilde u=\frac{\beta_0u}{\beta_0'}-\frac{1}{2}\frac{\beta_1}{\beta_0'}\partial_xu-\frac{1}{4}\frac{\beta_1^2}{\beta_0'}u^2-\frac{\beta_1}{2\beta_0'}\partial_x^{-1}\partial_yu \tag{6.9}$$

as the gauge transformation from general gauge $(V_0=\beta_0u,V_1=\beta_1u)$ to the other gauge $(\tilde V_0=\beta_0'\tilde u,\ \tilde V_1=0)$. For $\beta_0=0$ (6.9) is the two dimensional Miura transformation [21].

In the gauge $V_0=\beta_0u,\ V_1=\beta_1u$ equation (6.8) gives

$$\Big(\beta_0-\frac{\beta_1}{2}\partial_x-\frac{\beta_1^2}{2}u-\frac{\beta_1}{2}\partial_x^{-1}\partial_y\Big)\Big\{\frac{\partial u}{\partial t}-\omega_{10}\partial_xu-\omega_{11}\Big[\frac{1}{4}\partial_x^3u+ \tag{6.10}$$

$$+\frac{3}{2}\beta_0u\partial_xu-\frac{3}{8}\beta_1^2u^2\partial_xu+\frac{3}{4}\partial_x^{-1}\partial_y^2u-\frac{3}{4}\beta_1(\partial_xu)(\partial_x^{-1}\partial_yu)\Big]\Big\}=0.$$

From (6.10) we have the two-dimensional Gardner equation

$$\frac{\partial u}{\partial t}=\omega_{10}\partial_xu+\frac{1}{4}\omega_{11}\big[\partial_x^3u+6\beta_0u\partial_xu- \tag{6.11}$$

$$-\frac{3}{2}\beta_1^2u^2\partial_xu+3\partial_x^{-1}\partial_y^2u-3\beta_1(\partial_xu)(\partial_x^{-1}\partial_yu)\big].$$

At $\beta_0=1$, $\beta_1=0$ equation (6.11) converts to (6.6) and for $\beta_0=0$, $\beta_1=1$ it coincides with (6.7). Backlund transformations for equation (6.11) can be obtained from the gauge invariant part (6.4) of Backlund transformation (6.3). Putting $V_0=\beta_0u,\ V_1=\beta_1u,\ V_0'=\beta_0u',\ V_1'=\beta_1u'$, from (6.4) we obtain

$$\Big[2b_{00}(W_0'-W_0)+b_{10}(\partial_x(W_0'+W_0))+b_{10}(W_0'-W_0)\int_{-\infty}^{x}(W_0'-W_0)-$$

$$-b_{10}\int_{-\infty}^{x}((\partial_yW_0')-(\partial_yW_0))\Big]\exp\Big(-\frac{1}{2}\int_{-\infty}^{x}(V_1-V_1')\Big)=\Big[\beta_0-\frac{\beta_1}{2}\partial_x-\frac{\beta_1^2}{2}u-\frac{\beta_1}{2}\partial_x^{-1}\partial_y\Big]\times \tag{6.12}$$

$$\times\Big\{2b_{00}(u'-u)+b_{10}(\partial_x(u'+u))-b_{10}\int_{-\infty}^{x}((\partial_yu')-(\partial_yu))+$$

$$+b_{10}(u'-u)\int_{-\infty}^{x}\Big[\beta_0(u'-u)-\frac{\beta_1^2}{4}(u'^2-u^2)-\frac{\beta_1}{2}\partial_x^{-1}((\partial_yu')-(\partial_yu))\Big]\Big\}\exp\Big(-\frac{\beta_1}{2}\int_{-\infty}^{x}(u-u')\Big)=0.$$

As a result we have Backlund transformation for Gardner equation (6.11):

$$2b_{00}(u'-u)+b_{10}\partial_x(u'+u)-b_{10}\int_{-\infty}^{x}(\partial_y u'-\partial_y u)+$$
$$+b_{10}(u'-u)\int_{-\infty}^{x}[\beta_0(u'-u)-\frac{\beta_1^2}{4}(u'^2-u^2)-\frac{\beta_1}{2}\partial_x^{-1}(\partial_y u'-\partial_y u)]=0. \qquad (6.13)$$

At $\beta_0=1$, $\beta_1=0$ (6.13) is Backlund transformation for KP equation (6.6) and for $\beta_0=0$, $\beta_1=1$ it is Backlund transformation for modified KP equation (6.7).

In conclusion note that in frame of standard version of the inverse scattering transform method [1] general system of equations (6.5) is the commutativity condition $T_1T_2-T_2T_1=0$ of the following two operators

$$T_1=\partial_x^2+V_1\partial_x+V_0+\partial_y,$$
$$T_2=-\omega_{11}(t)[4\partial_x^3+6V_1\partial_x^2+(3(\partial_x V_1)+\frac{3}{2}V_1^2-3(\partial_x^{-1}\partial_y V_1)+$$
$$+6V_0-\omega_{10}(t))\partial_x+4(\partial_x V_0)+2V_0V_1-\varphi]+\partial_t .$$

VII. Examples: N = 3

Here we present some examples of integrable equations and Backlund transformations for N = 3 in the gauge $V_2=0$.

Operators ℓ_k^+ at $V_2=0$ are of the form

$$\ell_1^+=-\partial_y+V_0'-V_0-V_1\partial_x-\partial_x^3,\ \ell_2^+=V_1'-V_1-3\partial_x^2,\ \ell_3^+=-3\partial_x . \qquad (7.1)$$

Gauge invariants are

$$W_0=V_0-\frac{1}{3}\partial_x^2V_2-\frac{1}{3}V_1V_2+\frac{2}{27}V_2^3-\frac{1}{3}\partial_x^{-1}\partial_y V_2,$$
$$W_1=V_1-\partial_x V_2-\frac{1}{3}V_2^2. \qquad (7.2)$$

Backlund transformation (4.10) with nonzero b_{00}, b_{10}, b_{20} and all the rest $b_{kn}=0$ is of the form

$$b_{00}(V_0'-V_0)+b_{10}[\partial_x V_0'-\frac{1}{3}\partial_x^2(V_1'-V_1)-\frac{1}{3}V_1(V_1'-V_1)+$$
$$+\frac{1}{3}(V_0'-V_0)\partial_x^{-1}(V_1'-V_1)-\frac{1}{3}\partial_x^{-1}\partial_y(V_1'-V_1)]+ \qquad (7.3)$$

$$+b_{20}[-\frac{1}{3}\partial_x^3(V_1'+V_1)+\frac{2}{3}\partial_x^2V_0'+\frac{1}{3}\partial_x^2V_0+\frac{1}{3}V_1'(V_0'-V_0)-$$
$$-\frac{1}{3}V_1\partial_x(V_1'+V_1)-\frac{1}{3}(V_1'-V_1)\partial_x(V_1'-V_1)+\frac{1}{3}(\partial_x V_0')\partial_x^{-1}(V_1'-V_1)+$$
$$+\frac{1}{3}(V_0'-V_0)\partial_x^{-1}(V_0'-V_0)+\frac{1}{18}(V_0'-V_0)(\partial_x^{-1}(V_1'-V_1))^2-\frac{1}{9}V_1(V_1'-V_1)\partial_x^{-1}(V_1'-V_1)-$$
$$-\frac{1}{9}(\partial_x^2(V_1'-V_1))\partial_x^{-1}(V_1'-V_1)-\frac{1}{3}\partial_y(V_1'+V_1)-\frac{1}{3}\partial_x^{-1}(\partial_y V_0'-\partial_y V_0)+$$
$$+\frac{1}{18}\partial_y(\partial_x^{-1}(V_1'-V_1))^2]=0, \qquad (7.3)$$

$$b_{00}(V_1'-V_1)+b_{10}[V_0'-V_0+\partial_x V_1+\frac{1}{3}(V_1'-V_1)\partial_x^{-1}(V_1'-V_1)]+$$
$$+b_{20}[-\frac{1}{3}\partial_x^2V_1'-\frac{2}{3}\partial_x^2V_1+\partial_x(V_0'+V_0)+\frac{1}{3}V_1(V_1'-V_1)+$$
$$+\frac{1}{3}(\partial_x V_1)\partial_x^{-1}(V_1'-V_1)+\frac{1}{3}(V_0'-V_0)\partial_x^{-1}(V_1'-V_1)+$$
$$+\frac{1}{3}(V_1'-V_1)\partial_x^{-1}(V_0'-V_0)+\frac{1}{18}(V_1'-V_1)(\partial_x^{-1}(V_1'-V_1))^2-\frac{1}{3}\partial_x^{-1}(\partial_y(V_1'-V_1))]=0.$$

Transformation (7.3) is Backlund transformation for any equation (4.16) in the gauge $V_2=0$.

Present now some integrable equations. Let in the system (4.16) only functions ω_{10}, ω_{11}, ω_{20} and ω_{21} are not equal to zero. Calculating $\mathcal{L}^+_{(1,0)(3)}$, $\mathcal{L}^+_{(1,1)(3)}$, $\mathcal{L}^+_{(2,0)(3)}$, $\mathcal{L}^+_{(2,1)(3)}$ by the formula (4.17), we obtain in this case the following system of two equations

$$\frac{\partial V_0}{\partial t}=\omega_{10}(t)\partial_x V_0+\omega_{11}(t)[-\frac{2}{9}\partial_x^5V_1+\frac{1}{3}\partial_x^4V_0+\frac{2}{3}\partial_x(V_1\partial_x V_0)+$$
$$+\frac{2}{3}\partial_x(V_0^2)-\frac{2}{3}V_1\partial_x^3V_1-\frac{4}{3}(\partial_x V_1)(\partial_x^2V_1)-\frac{4}{9}V_1^2\partial_x V_1- \qquad (7.4)$$
$$-\frac{2}{3}\partial_x\partial_y V_0-\frac{1}{9}\partial_x^2\partial_y V_1-\frac{1}{3}V_1\partial_y V_1+\frac{1}{9}\partial_x^{-1}\partial_y^2V_1]+$$

$$+\omega_{20}(t)[\partial_x^2 V_0 - \tfrac{2}{3}\partial_x^3 V_1 - \tfrac{2}{3}V_1\partial_x V_1 - \tfrac{2}{3}\partial_y V_1] +$$

$$+\omega_{21}(t)[-\tfrac{1}{9}\partial_x^5 V_0 - \tfrac{5}{9}\partial_x^3(V_1V_0) - \tfrac{5}{9}\partial_x(V_0\partial_x^2 V_1) + \tfrac{5}{3}\partial_x(V_0\partial_x V_0) -$$

$$-\tfrac{5}{9}\partial_x(V_0V_1^2) - \tfrac{5}{9}\partial_x^2\partial_y V_0 + \tfrac{5}{9}\partial_x^{-1}\partial_y^2 V_0 - \tfrac{10}{9}V_0(\partial_y V_1) -$$

$$-\tfrac{5}{9}V_1\partial_y V_0 - \tfrac{5}{9}(\partial_x V_0)\partial_x^{-1}\partial_y V_1],$$

$$\frac{\partial V_1}{\partial t} = \omega_{10}(t)\partial_x V_1 + \omega_{11}(t)[-\tfrac{1}{3}\partial_x^4 V_1 + \tfrac{2}{3}\partial_x^3 V_0 + \tfrac{4}{3}\partial_x(V_0V_1) -$$

$$-\tfrac{2}{3}\partial_x(V_1\partial_x V_1) - \tfrac{4}{3}\partial_y V_0 - \tfrac{1}{3}\partial_x\partial_y V_1] + \omega_{20}(t)[2\partial_x V_0 - \partial_x^2 V_1] +$$

$$+\omega_{21}(t)[-\tfrac{1}{9}\partial_x^5 V_1 - \tfrac{5}{9}\partial_x(V_1\partial_x^2 V_1) - \tfrac{5}{3}\partial_x(V_0\partial_x V_1) - \tfrac{5}{9}V_1^2\partial_x V_1 +$$

$$+\tfrac{5}{3}\partial_x(V_0^2) - \tfrac{5}{9}\partial_x^2\partial_y V_1 + \tfrac{5}{9}\partial_x^{-1}\partial_y^2 V_1 - \tfrac{5}{9}V_1\partial_y V_1 - \tfrac{5}{9}(\partial_x V_1)\partial_x^{-1}\partial_y V_1].$$

There are some interesting particular cases of the system (7.4). For $\omega_{20} = const$ and $\omega_{10} = \omega_{11} = \omega_{21} = 0$ from (7.4) we obtain

$$\frac{\partial V_0}{\partial t} = \omega_{20}[\partial_x^2 V_0 - \tfrac{2}{3}\partial_x^3 V_1 - \tfrac{2}{3}V_1\partial_x V_1 - \tfrac{2}{3}\partial_y V_1],$$
$$\frac{\partial V_1}{\partial t} = \omega_{20}[2\partial_x V_0 - \partial_x^2 V_1]. \qquad (7.5)$$

It is easy to see that the system (7.5) is equivalent at $\omega_{20} = i\sqrt{\frac{3}{2}}$ to the equation

$$\frac{\partial^2 V_1}{\partial t^2} = \frac{1}{2}\partial_x^4 V_1 + \partial_x^2 V_1^2 + 2\partial_x\partial_y V_1 \qquad (7.6)$$

that is the twodimensional generalization of Boissinesq equati-

on. However compare (7.6) with (6.6) at $\omega_{10} = 0$ we see that equation (7.6) is nothing but KP equation (6.6) with the substitution $t \leftrightarrow y$.

For $\omega_{10} = \omega_{11} = \omega_{20}$ the system (7.4) is the following one

$$\frac{\partial V_0}{\partial t} = \omega_{21}[-\tfrac{1}{9}\partial_x^5 V_0 - \tfrac{5}{9}\partial_x^3(V_1V_0) - \tfrac{5}{9}\partial_x(V_0\partial_x^2 V_1) +$$
$$+\tfrac{5}{3}\partial_x(V_0\partial_x V_0) - \tfrac{5}{9}\partial_x(V_0V_1^2) - \tfrac{5}{9}\partial_x^2\partial_y V_0 + \tfrac{5}{9}\partial_x^{-1}\partial_y^2 V_0 - \qquad (7.7)$$
$$-\tfrac{10}{9}V_0(\partial_y V_1) - \tfrac{5}{9}V_1\partial_y V_0 - \tfrac{5}{9}(\partial_x V_0)\partial_x^{-1}\partial_y V_1],$$
$$\frac{\partial V_1}{\partial t} = \omega_{21}[-\tfrac{1}{9}\partial_x^5 V_1 - \tfrac{5}{9}\partial_x(V_1\partial_x^2 V_1) - \tfrac{5}{3}\partial_x(V_0\partial_x V_1) - \tfrac{5}{9}V_1^2\partial_x V_1 +$$
$$+\tfrac{5}{3}\partial_x(V_0^2) - \tfrac{5}{9}\partial_x^2\partial_y V_1 + \tfrac{5}{9}\partial_x^{-1}\partial_y^2 V_1 - \tfrac{5}{9}V_1\partial_y V_1 - \tfrac{5}{9}(\partial_x V_1)\partial_x^{-1}\partial_y V_1].$$

System of equations (7.7) is the twodimensional generalization of the system considered in [16]. System (7.7) permits the reductions $V_0 = 0$ and $V_0 = \frac{1}{2}\partial_x V_1$. In the case $V_0 = 0$ and $\omega_{21} = -9$ the system (7.7) is reduced to equation

$$\frac{\partial V_1}{\partial t} = \partial_x^5 V_1 + 5\partial_x(V_1\partial_x^2 V_1) + 5V_1^2\partial_x V_1 + 5\partial_x^2\partial_y V_1 -$$
$$-5\partial_x^{-1}\partial_y^2 V_1 + 5V_1\partial_y V_1 + 5(\partial_x V_1)(\partial_x^{-1}\partial_y V_1), \qquad (7.8)$$

that is the twodimensional generalization of Sawada-Kotera equation [22, 16]. Under the reduction $V_0 = \frac{1}{2}\partial_x V_1$ and at $\omega_{21} = -9$ system (7.7) is equivalent to

$$\frac{\partial V_1}{\partial t} = \partial_x^5 V_1 + 5V_1\partial_x^3 V_1 + \tfrac{25}{2}(\partial_x V_1)(\partial_x^2 V_1) + 5V_1^2\partial_x V_1 + \qquad (7.9)$$
$$+5\partial_x^2\partial_y V_1 - 5\partial_x^{-1}\partial_y^2 V_1 + 5V_1\partial_y V_1 + 5(\partial_x V_1)(\partial_x^{-1}\partial_y V_1),$$

that is the twodimensional generalization of Kupershmidt equation [16]. Note that the parts of equations (7.8) and (7.9) which contain the terms with derivative $\frac{\partial}{\partial y}$ coincide.

It is easy also to see that all equations (7.4)-(7.9) can be represented in the local form by introducing the potentials W_i $(V_i = \partial_x W_i)$.

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ГРУППА БЭКЛУНД-КАЛОДЖЕРО И ОБЩАЯ СТРУКТУРА ИНТЕГРИРУЕМЫХ УРАВНЕНИЙ ДЛЯ ДВУМЕРНОЙ ЗАДАЧИ ГЕЛЬФАНДА-ДИКОГО-ЗАХАРОВА-ШАБАТА. БИЛОКАЛЬНЫЙ ПОДХОД

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