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ON SOME PROPERTIES OF THE $\mathbf{SU(N_c)}$ YANG-MILLS VACUUM STATE

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On some properties of the $SU(N_c)$ Yang-Mills vacuum state

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Аннотация

The asymptotic behaviour of the vacuum energy density, $\overline{E}_{vac}(\theta)$, at $\theta \to \pm i\infty$ is found out. A new interpretation and a qualitative discussion of the $\overline{E}_{vac}(\theta)$ behaviour are presented. It is emphasized that the vacuum is doubly degenerate at $\theta = \pi$, and the quark electric string can terminate on the domain wall interpolating between these two vacua. The potential of the monopole field condensing in the Yang-Mills vacuum is obtained.

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1. Let us consider the (Euclidean) YM $SU(N_c)$ theory defined by the integral:

$$Z = \sum_k \int dA_\mu \, \delta(Q-k) \exp\left\{ - \int dx \, I_o(x) \right\} , \qquad I_o = \frac{1}{4g_o^2} \, G_{\mu\nu}^2 \, , \tag{1}$$

where Q is the topological charge, and let us integrate out the gluon fields with the constraint that the composite field $G_{\mu\nu}^2$ is fixed: [1]

$$Z = \int dS \, \theta(S) \exp\left\{ - \int dx \, I(S) \right\} , \quad I(S) = N_c^2 \frac{b_o}{4} \left[S \ln \frac{M_o^4}{\bar{\Lambda}^4} + f(S, M_o^2) \right] ,$$

$$\exp\left\{ - \int dx \, N_c^2 \frac{b_o}{4} f(S, M_o^2) \right\} = \sum_k \int dA_\mu \, \delta(Q-k) \;\; \delta\left(S - \frac{G_{\mu\nu}^2}{32\pi^2 N_c} \right) , \tag{2}$$

where M_o is the ultraviolet cut off, and we have substituted: $b_o = 11/3$, $1/g_o^2 = (\, N_c \, b_o / 32\pi^2 \,) \ln \left(M_o^4 / \bar{\Lambda}^4 \right)$.

Let us emphasize that the function $f(S, M_o^2)$ does not know about $\bar{\Lambda}$. Thus, recalling for a renormalizability of the theory, it can be written in the form:

$$f(S, M_o^2) = -S \ln \left(\frac{M_o^4}{S} \right) + S \, C_N + \frac{1}{S^{3/2}} \partial_\mu S \, \Delta_o \left(\frac{q^2 = -\partial^2}{S^{1/2}} \right) \partial_\mu S +$$

$$+ O\left((\partial_\mu S)^4 \right) + \cdots + \frac{1}{N_c^2} f_s(S, M_o^2) \, . \tag{3}$$

So, the partition function takes the form:

$$Z = \int dS \;\; \theta(S) \exp\left\{ - \int dx \, N_c^2 \frac{b_o}{4} \left[L_o(S) + S \, C_N + \frac{1}{N_c^2} f_s(S, M_o^2) \right] \right\} ,$$

[1] This is what differs our approach from those proposed by G. Savvidy [1], the latter one leading to instabilities.

$$L_o(S) = S\,\ln\left(\frac{S}{\bar{\Lambda}^4}\right) + \frac{1}{S^{3/2}}\partial_\mu S\,\Delta_o\left(\frac{q^2}{S^{1/2}}\right)\partial_\mu S + O\left((\partial_\mu S)^4\right) + \cdots \qquad (4)$$

Because we have not integrated yet the quantum loop contributions of the field S, the action still depends (logariphmically) on the cut off M_o^2 (through the function f_s in eq.(4)), and this dependence will disappear only after the quantum loop corrections of the fields S will be accounted for completely. However, these quantum corrections are amenable to the standard $1/N_c$ - counting. And because the field S is only one out of $\sim N_c^2$ degrees of freedom, the $1/N_c$ - counting implies that its quantum fluctuations will give only $1/N_c^2$ - corrections to the action. [2] This means, in particular, that if we neglect the quantum loop contributions of the field S altogether (so that the field S will become a free quantum field), then the above action in eq.(4) will be a generating functional of the 1-PI (one particle irreducible) Green functions of the field S (and will differ from the exact generating functional by the $1/N_c^2$ corrections only). For instance, the potential (i.e. without space-time derivatives) term will be: $U = N_c^2\frac{b_o}{4}S\,[\ln\left(\frac{S}{\bar{\Lambda}^4}\right) + C_N\,]$, etc.

Let us emphasize now that, on the one hand, the constant C_N in eq.(4) can be considered as the external (constant) source of the field S and, on the other hand, all the dependence on this source can be absorbed into a redefinition of the scale parameter $\bar{\Lambda}$. For our further purposes, let us substitute it (temporarily) by the local source, $C_N(x)$.

Now, accounting for the quantum loop contributions of the field S and considering $C_N(x)$ as a source, the partition function can be written in the form of the Legendre fransform:

$$Z = \oint dS\,\theta(S)\,\exp\left\{-\int dx\,N_c^2\frac{b_o}{4}\left[L_o(S) + \frac{1}{N_c^2}\,\delta L(S) + S\,C_N(x)\right]\right\}, \qquad (5)$$

where the integral $\oint dS$ in eq.(5) means that all quantum loop corrections are ignored now, so that $L = (\,L_o + (1/N_c^2)\,\delta L\,)$ is the exact generating functional of the 1-PI Green functions of the field S. Finally, using the above described property that all the dependence on the source $C_N(x)$ (when it becomes constant) can be absorbed into a redefinition of $\bar{\Lambda}$, its potential part becomes completely fixed, and so L can be written in the form ($B_N = 1 + O(1/N_c^2)$ is another constant):

$$L = \left[S\ln\left(\frac{S}{B_N\bar{\Lambda}^4}\right) + \frac{1}{S^{3/2}}\partial_\mu S\,\Delta_1\left(\frac{q^2}{S^{1/2}}, N_c\right)\partial_\mu S + O\left((\partial_\mu S)^4\right) + \cdots\right] \qquad (6)$$

[2] In all the above formulae the leading N_c -dependence is written out explicitly, while the dependence on the nonleading corrections in powers of $1/N_c$ is implicit.

where, for instance, the function Δ_1 approaches Δ_o at $N_c \gg 1$, etc.

Let us emphasize that, as it is seen from its derivation, the action $L(S)$ in eqs.(5,6) is not some approximate effective or low energy action but the exact one, and the exact answers for all correlators of the field S can be obtained from this action taking derivatives over the source and calculating the tree diagrams only (as all quantum loop contributions of the field S were accounted for already). [3]

The above described way of obtaining the exact generating functional of the 1-PI Green functions of composite fields will be widely used below and in the subsequent paper [2]. In a few words, it can be formulated as follows. Let us suppose that we have some quantum field ϕ (elementary or composite) with the (Euclidean) partition function:

$$Z = \int d\phi \ \exp\left\{ - \int dx \ [\, L_o(\phi) + J(x)\phi \,] \right\} , \tag{7}$$

where $J(x)$ is a source. After integrating out all quantum loop effects, it can be represented in the form of the Legendre transform:

$$Z = \oint d\phi \exp\left\{ - \int dx \ [\, L_o(\phi) + \Delta L(\phi) + J(x)\phi \,] \right\} , \tag{8}$$

where $L(\phi) = L_o(\phi) + \Delta L(\phi)$ is the exact generating functional of the 1-PI Green functions of the quantum field ϕ, and the exact correlators of the field ϕ are obtained from eq.(8) using the tree diagrams only. We would like to emphasize here that, clearly, $L(\phi)$ is independent of the source $J(x)$. So, we can freely change the source or even to introduce it afterwards, - $L(\phi)$ (considered as a function of ϕ and its derivatives) will stay intact. And we will widely use this property below and in [2].

Little is known about the inverse propagator of the S-field, except what can be obtained from the asymptotic freedom and operator expansions. Let us define the propagator: $i \int dx \exp(iqx) \langle S(x) S(0) \rangle_{con} = D(q^2, \Lambda^2)$. At $q^2 \gg \Lambda^2$:

$$D(q^2, \Lambda^2) = \left\{ \left[C_1 \, \alpha_s(\frac{q^2}{\Lambda^2}) \, q^4 \, (\, 1 + O(\alpha_s \,)) \right] + \left[C_2 \, \alpha_s(\frac{q^2}{\Lambda^2}) \, \Lambda^4 \, (\, 1 + O(\alpha_s)) \right] + \right.$$

[3] To simplify, we used the one-loop β-function in the above equations, neglecting the terms with the nonleading dependence on the cut off, M_o. More precisely, to obtain the renormalization group invariant expressions we have to fix the renormalization group invariant quantities, e.g. $(-1/N_c^2 b_o) T_{\mu\mu} = z_G(g_o^2) S = (1 + O(g_o^2)) S$ in this case, where $T_{\mu\mu}$ is the energy-momentum tensor trace. Having this in mind, it is implied everywhere below that appropriate renormalization factors, z_i, are included already into definitions of all fields and parameters we deal with.

$$+\left[C_3\,\alpha_s^{\delta}(\frac{q^2}{\Lambda^2})\frac{\Lambda^6}{q^2}\,(\,1+O(\alpha_s))\right]+\cdots\Bigg\}\,.$$

Clearly, the behaviour of $\Delta_1(q^2/S^{1/2},N_c)$ in eq.(6) at $q^2 \gg S^{1/2}$ can be reconstructed from the above behaviour of D, with a replacement: $\Lambda^2 \to [eS]^{1/2}$. [4]

In what follows we will be interested mainly in the potential part (i.e. without space-time derivatives) of the action and will ignore all terms with such derivatives.

It follows from eqs.(5,6) that the field S condenses in the YM-vacuum, and the values of the condensate and the vacuum energy density are:

$$\overline{S}=e^{-1}\,\Lambda^4\,,\quad \overline{E}_{vac}=-N_c^2\frac{b_o}{4}\,\Lambda^4\;<\;0\,. \tag{9}$$

As it is, the potential in eq.(6) is well known (see e.g. [3]). However, the field S was considered previously mainly as the effective dilaton field, i.e. the interpolating field of the lightest scalar gluonium. Besides, such Lagrangeans were usually considered as some "effective" Lagrangeans, the meaning of "effective" was obscure, as well as their connection with the original YM-Lagrangean. Our approach allows to elucidate its real origin and meaning and, on this basis, to use it for the investigation of the vacuum and correlators properties.

2. Let us extend now our theory and add the θ-term to the action I_o in eq.(1), and let us integrate out the gluon fields with the fields S(x) and P(x) both fixed ($\tilde{\theta}=i\theta$, $Q=N_c\int dx P$):

$$Z=\sum_k\int dS\,\theta(S)\,\int dP\,\theta(S-|P|)\,\delta(Q-k)$$

$$\exp\left\{-\int dx\,N_c^2\left[\frac{b_o}{4}S\ln\frac{M_o^4}{\Lambda^4}+S\,I_o(\frac{P}{S},\frac{M_o^4}{S})-\frac{\tilde{\theta}}{N_c}P\right]\right\}\,,$$

$$\exp\left\{-\int dx\,N_c^2\,S\,I_o(\frac{P}{S},\frac{M_o^4}{S})\right\}=$$

$$=\int dA_\mu\,\delta\left(S-\frac{G_{\mu\nu}^2}{32\pi^2N_c}\right)\,\delta\left(P-\frac{G\tilde{G}}{32\pi^2N_c}\right)\,. \tag{10}$$

[4] As can be easily checked, the behaviour of $\Delta_o(q^2/S^{1/2})$ at large q^2 agrees precisely with that the quantum loop corrections of the field S give only $\sim(1/N_c)\ln M_o^2$ contribution to the renormalization of the charge $1/g_o^2$, i.e. a relative $\sim 1/N_c^2$ correction.

The positivity of the Euclidean integration measure (at real positive $\tilde{\theta}$) leads to a number of useful sign inequalities, like $\overline{S}(\tilde{\theta}) \geq \overline{P}(\tilde{\theta}) \geq 0$, etc. Because $\overline{P}(\tilde{\theta}) = (b_o/4)\, d\overline{S}(\tilde{\theta})/d\tilde{\theta}$, this shows that $\overline{S}(\tilde{\theta})$ grows monotonically with $\tilde{\theta}$ (so that the energy density decreases monotonically with $\tilde{\theta}$).

Because the θ - term in the Lagrangean can be considered as a source of the field P (moreover, we can even replace it by the local function, $\theta(x)$), integrating out the quantum loop contributions of the S and P fields and using the above described considerations, the partition function can be rewritten in the form:

$$Z = \sum_k \oint dS\, \theta(S) \oint dP\, \theta(S - |P|)\, \delta(Q - k) \times$$

$$\times \exp\left\{ - \int dx N_c^2 \left[I(S,P) - \frac{\tilde{\theta}}{N_c} P \right] \right\}, \tag{11}$$

Let us write the general form of the potential in eq.(11):

$$U(S,P) = \frac{b_o}{4} S \left[\ln\left(\frac{S}{\Lambda^4}\right) + f(z = \frac{P}{S}) \right] - \frac{\tilde{\theta}}{N_c} P\,, \tag{12}$$

and the stationary point equations:

$$\frac{\tilde{\theta}}{N_c} = \frac{b_o}{4} f'(\bar{z}), \quad \frac{b_o}{4} \left[\ln\left(\frac{e\overline{S}}{\Lambda^4}\right) + f(\bar{z}) \right] = \frac{\tilde{\theta}}{N_c} \bar{z}. \tag{13}$$

Consider now the behaviour at $\tilde{\theta} \to \infty$. It is seen from eq.(13) that $\bar{z}(\tilde{\theta})$ approach z_o, - the singularity point of $f'(z)$. It looks physically unacceptable if $f(z)$ had singularities (say, poles or branch points) inside the physical region $0 < z < 1$. They can develop at the edges of the physical region only: $z \to 0$ or $z \to 1$. The behaviour of $f(z)$ at $z \to 0$ is regular however, $\sim z^2$. So, we conclude that: $\bar{z}(\tilde{\theta}) = \overline{P}(\tilde{\theta})/\overline{S}(\tilde{\theta}) \to 1$ at $\tilde{\theta} \to \infty$. Supposing that the singularity is gentle enough so that $f(1)$ is finite, one obtains then from eq.(13):

$$\overline{P}(\tilde{\theta}) \to \overline{S}(\tilde{\theta}) \to C \exp\left\{ \frac{n}{b_o} \frac{\tilde{\theta}}{N_c} \right\}, \; C = e^{-f(1)-1} \Lambda^4, \quad \tilde{\theta} \to \infty, \tag{14}$$

where $n = 4$ is the space-time dimension (compare with the CP^N model, see appendix).

To illustrate the typical properties, let us write the simple model for $U(S,P)$ in eq.(12). It looks (in Minkowsky space) as follows:

$$U(S,P) = \frac{1}{2}\left\{(S+iP)\ln\left(\frac{S+iP}{\Lambda^4 e^{i\theta/N_c}}\right) + h.c.\right\} - \frac{1}{12}S\ln\left(\frac{S}{\Lambda^4}\right). \qquad (15)$$

When integrated over S and P fields it gives (at the space-time volume $V \to \infty$):

$$\overline{S}(\theta) = e^{-1}\Lambda^4\left[\cos^{4/b_o}\left(\frac{\theta}{N_c}\right)\right]_{2\pi},$$

$$\overline{E}_{vac} = -N_c^2\frac{b_o}{4}\overline{S}(\theta), \quad \overline{P}(\theta) = -\frac{1}{N_c}\frac{d\overline{E}_{vac}(\theta)}{d\theta}. \qquad (16)$$

Here the notation $[f(\theta/N_c)]_{2\pi}$ means that this function is $f(\theta/N_c)$ at $-\pi \le \theta \le \pi$, and is glued then to be periodic in $\theta \to \theta + 2\pi k$, i.e.:

$$\left[f(\frac{\theta}{N_c})\right]_{2\pi} = \min_k\ f\left(\frac{\theta+2\pi k}{N_c}\right). \qquad (17)$$

Evidently, the above periodically glued structure of the vacuum energy density results from the quantization of the topological charge in our theory, and is not connected with a concrete form of the action $I(S,P)$ in eq.(11).

The above model does not pretend to be the exact answer, but it is simple, has a reasonable qualitative behaviour in the whole complex plane of θ, and obeys the right asymptotic behaviour at $\theta \to \pm i\infty$.

It was inspired by a "different status" of two parts in $b_o/4 = (1-\frac{1}{12})$. The first part is connected with zero mode contributions in the instanton background, and the analytic in $\chi = (S+iP)$ term in eq.(15) is expected to be connected with this part of b_o only, while the second, nonanalytic in χ, part of eq.(15) is expected to be due to the "nonzero mode part" of b_o only.

In Euclidean space, one has from eq.(16):

$$\overline{S}(\tilde{\theta}) = e^{-1}\Lambda^4\ \mathrm{ch}^{4/b_o}\left(\frac{\tilde{\theta}}{N_c}\right), \quad \overline{P}(\tilde{\theta}) = \mathrm{th}\left(\frac{\tilde{\theta}}{N_c}\right)\overline{S}(\tilde{\theta}), \qquad (18)$$

so that the asymptotic behaviour, eq.(14), is reproduced, and $\bar{z}(\tilde{\theta}) = \overline{P}/\overline{S} \to 1$ at large $\tilde{\theta}$. In principle, the equation (18) can be checked in lattice calculations. [5]

[5] There is another simple model: $(1/N_c^2)U = \frac{1}{2}\left\{\phi\ln\frac{\phi}{\Lambda^4 e^{i\theta/N_c}} + h.c.\right\}$, $\phi = (b_o/4)S +$

One point is worth mentioning in connection with the asymptotic behaviour, eq.(14). Based on the quasiclassical (one loop) instanton calculations [4], one would expect the following qualitative picture. In $SU(N_c)$, each instanton splits up into N_c "instantonic quarks" which appear as appropriate degrees of freedom in the dense instanton ensemble. As a result, each instantonic quark carries the factor $\exp\{i\theta/N_c\}$ in its density. So, one would expect the behaviour $\bar{P}(\tilde{\theta}) \to \bar{S}(\tilde{\theta}) \sim \exp\{\tilde{\theta}/N_c\}$ at large $\tilde{\theta}$, in disagreement with eq.(14). We conclude that something is missing here in the above picture of instantonic quarks, even in its θ-dependence.

3. We will describe now a new qualitative interpretation of (glued) periodicity properties of the vacuum energy density, $\overline{E}_{vac}(\theta)$. Let us consider first $N_c = 2$ for simplicity, and let us suppose the "standard" picture of the confinement mechanism to be valid. I.e., the internal (dynamical) Higgs breaking $SU(2) \to U(1)$ takes place and, besides, the $U(1)$ magnetic monopoles condense. As shown by E. Witten [5], the monopoles turn into the dyons with the $U(1)$ electric charge $\theta/2\pi$ when the θ - term is introduced. Although, strictly speaking, the Witten result was obtained in the quasiclassical region only, because the effect is of a qualitative nature there are all the reasons to expect it will survive in the strong coupling region also (and only the units of the electric and magnetic charges will change).

So, there are pure monopoles and antimonopoles with the magnetic and electric charges $(g,\, e) = (1,\, 0)$ and $(\bar{g},\, \bar{e}) = (-1,\, 0)$ in the condensate at $\theta = 0$, while they turn into the dyons and antidyons with the charges $d_1^\theta = (1,\, \theta/2\pi)$ and $\bar{d}_1^\theta = (-1,\, -\theta/2\pi)$ respectively, as θ starts to deviate from zero. As a result, the vacuum energy density begins to encrease, as it follows from general considerations.

It is a specific property of our system that there are two types of condensates made of the dyons and antidyons with the charges: $\{(1, 1/2); (-1, -1/2)\}$ and $\{(1, -1/2); (-1, 1/2)\}$, and having the same energy density. [6] Besides,

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iP, leading to $\overline{E}_{vac}(\theta) \sim [\cos(\frac{b_o}{4}\frac{\theta}{N_c})]_{2\pi}$, i.e. with the same asymptotic behaviour at $\theta \to \pm i\infty$. The problem with it is that there are reasons to expect that the vacuum energy is exactly zero at $\theta = \pi$ and $N_c = 2$, and this model does not fulfil this, while eq.(15) does. Besides, we see no reasons here for the potential to be analytic function of one variable.

[6] This can be seen, for instance, as follows. Let us start from the pure monopole condensate at $\theta = 0$ and let us move along the path: $\theta = 0 \to \theta = \pi$. The vacuum state will consist of $d_1(1, \frac{1}{2})$ - dyons and $\bar{d}_1(-1, -\frac{1}{2})$ - antidyons. Let us move now along the path: $\theta = 0 \to \theta = -\pi$. The vacuum state will consist now of $d_2(1, -\frac{1}{2})$ - dyons and $\bar{d}_2(-1, \frac{1}{2})$ - antidyons. Because the vacuum energy density is even under $\theta \to -\theta$, these two vacuum states are degenerate.

Let us emphasize that the existence of two vacuum states at $\theta = \pi$ does not follow from symmetry considerations alone (like $\overline{E}_{vac}(\theta) = \overline{E}_{vac}(-\theta)$ and $\overline{E}_{vac}(\theta) = \overline{E}_{vac}(\theta + 2\pi k)$).

these two states belong to the same world as they are reachable one from another through a barrier, because there are electrically charged gluons, $(0, \pm 1)$, in the spectrum which can recharge these $(1, \pm 1/2)$ - dyons into each other. In contrast, the two vacuum states, $|\theta\rangle$ and $|-\theta\rangle$ at $\theta \neq 0$, π are unreachable one from another and belong to different worlds, as there is no particles in the spectrum capable to recharge the $(1, \pm\theta/2\pi)$ - dyons into each other.

Thus, the vacuum becomes twice degenerate at $\theta = \pi$, so that the "level crossing" (in the form of rechargement: $\{d_1 = (1, 1/2), \bar{d}_1 = (-1, -1/2)\} \to \{d_2 = (1, -1/2), \bar{d}_2 = (-1, 1/2)\}$) can take place if this will lower the energy density. And indeed it lowers, and this leads to a casp in $\overline{E}_{vac}(\theta)$. At $\theta > \pi$, the vacuum is filled now with new dyons with the charges: $d_2^\theta = (1, -1 + \theta/2\pi)$, $\bar{d}_2^\theta = (-1, 1 - \theta/2\pi)$. As θ increases further, the electric charge of the d_2^θ -dyons decreases, and the vacuum energy density decreases with it. Finally, at $\theta = 2\pi$, the d_2^θ-dyons (which were the (1,-1)-dyons at $\theta = 0$) become pure monopoles, and the vacuum state becomes exactly as it was at $\theta = 0$, i.e. the same condensate of the pure monopoles and antimonopoles.

We empasize that, as it follows from the above picture, it is wrong to imagine the vacuum state at $\theta = 2\pi$ as, for instance, a condensate of the dyons with the charges (1,-1), degenerate in energy with the pure monopole condensate at $\theta = 0$. [7]

As for $SU(N_c > 2)$, the above described picture goes through without changes if we suppose that $SU(2) \to U(1)$ is replaced by $SU(N_c) \to U(1)^{N_c - 1}$, and there are $(N_c - 1)$ types of monopoles in the condensate, with the magnetic charges $(m^A)_i = \pm n_i^A$, $A = 1, ..., N_c - 1$, $i = 1, ..., N_c$. (For instance,

It is sufficient to give a counterexample. So, let us consider the Georgy-Glashow model, with the large Higgs vacuum condensate resulting in $SU(2) \to U(1)$. In this case, the θ-dependence of the vacuum energy density is due to a rare quasiclassical gas of instantons, and is $\sim \cos(\theta)$. All the above symmetry properties are fulfilled, but there is only one vacuum state at $\theta = \pi$.

[7] In this respect, the widely used terminology naming the two singularity points, $u = \pm\Lambda^2$, on the $\mathcal{N} = 2$ $SU(N_c = 2)$ SYM moduli space as those where monopoles and respectively dyons become massless, is not quite adequate. Indeed, let us start from the vacuum $u = \Lambda^2$ where, by definition, the massless particles are pure monopoles, and let us move, for instance, along a circle to the point $u = -\Lambda^2$. On the way, the former massless monopole increases its mass because it becomes the $d_1^\theta = (1, \theta/2\pi)$ - dyon, while the former massive $d_2^o = (1, -1)$ - dyon diminishes its mass as it becomes the $d_2^\theta = (1, -1 + \theta/2\pi)$ - dyon. When we reach the point $u = -\Lambda^2$, i.e. $\theta = 2\pi$, the former dyon becomes massless just because it becomes the pure monopole here. So, an observer living in the world with $u = -\Lambda^2$ will also see the massless monopoles (not dyons, and this is distinguishable by their Coulomb interactions), as those living in the world with $u = \Lambda^2$.

it has been proposed by G.'t Hooft [6] that the good candidates to condense are the "minimal monopoles" having two consecutive and opposite charges, i.e. $n_i^A = \delta_i^A - \delta_{i+1}^A$ in this case). Now, when the θ- term is introduced into the Lagrangean, each one of these $(N_c - 1)$ monopoles will turn into the dyon with the same type electric charge: $(e^A)_i = \pm(\theta/2\pi)\, n_i^A$. So, the above described picture of dyon rechargement will be applicable as well, resulting in a cusp in $\overline{E}_{vac}(\theta)$ at $\theta = \pi$.

In fact, for the above described mechanism to be operative, there is no need to trace the real dynamical picture underlying a cusped behaviour of $\overline{E}_{vac}(\theta)$ in the infinitesimal vicinity of $\theta = \pi$. Formally, it is sufficient to say: "there is a possibility for electrically charged degrees of freedom to rearrange themselves without changing the volume energy". However, it may be useful to have a more visible picture of how it can proceed. At least formally, it can be thought as a typical first order phase transition. In a space with the coherent condensate of the $d_1 = (1, 1/2)$ - dyons and $\bar{d}_1 = (-1, -1/2)$ - antidyons, there appears a critical bubble with the coherent condensate of the $d_2 = (1, -1/2)$ - dyons and $\bar{d}_2 = (-1, 1/2)$ - antidyons deep inside, and with a transition region surface (domain wall) through which the averaged densities of two type dyons interpolate smoothly. This bubble expands then over all the space through a rechargement process $d_1 + \bar{d}_1 \to d_2 + \bar{d}_2$ occuring on a surface. This rechargement can also be thought as going through a copious "production" of charged gluon pairs, so that the underlying processes will be: $[d_1 = (1, 1/2)] + [\bar{g} = (0, -1)] \to [d_2 = (1, -1/2)]$ and $[\bar{d}_1 = (-1, -1/2)] + [g = (0, 1)] \to [\bar{d}_2 = (-1, 1/2)]$.

Some analogy with the simplest Schwinger model may be useful at this point, in connection with the above described rechargement process. Let us consider first the pure QED_2 without finite mass charged particles, and let us put two infinitely heavy "quarks" with the charges $\pm\theta/2\pi$ (in units of some e_o) at the edges of our (infinite length) space. It is well known that this is equivalent to introducing the θ-angle into the QED_2 Lagrangean. As a result, there is the empty vacuum at $\theta = 0$, and the long range Coulomb "string" at $\theta \neq 0$. The vacuum energy density increases as: $E(\theta) = C_o e_o^2\, \theta^2$, $C_o = const$, at any $0 \leq \theta < \infty$.

Let us add now some finite mass, $m \gg e_o$, and of unit charge e_o field ψ to the Lagrangean. When there are no external charges, this massive charged field can be integrated out, resulting in a small charge renormalization. But when the above "quarks" are introduced, the behaviour of $E(\theta)$ becomes nontrivial: $E(\theta) = C_o e_o^2 \min_k (\theta + 2\pi k)^2$. So, $E(\theta) = C_o e_o^2\, \theta^2$ at $0 \leq \theta \leq \pi$, and $E(\theta) = C_o e_o^2\, (2\pi - \theta)^2)$ at $\pi \leq \theta \leq 2\pi$.

The reason is clear. The external "quark" charge becomes equal 1/2 at

$\theta = \pi$. At this point, a pair of ψ - particles is produced from the vacuum, and they separate so that to recharge the external "quarks": $\pm 1/2 \to \mp 1/2$. As a result of this rechargement, there appears a cusp in $E(\theta)$, and $E(\theta)$ begins to decrease at $\theta > \pi$, so that the former "empty" vacuum is reached at $\theta = 2\pi$.

Let us return however to our dyons. The above described picture predicts also a definite behaviour of the topological charge density, $\overline{P}(\theta)$. At $0 < \theta < \pi$, i.e. in the condensate of the $d_1^\theta = (1, \theta/2\pi)$ - dyons and $\bar{d}_1^\theta = (-1, -\theta/2\pi)$ - antidyons, the product of signs of the magnetic and electric charges is positive for both d_1^θ - dyons and $\bar{d}_1^\theta$ - antidyons. Thus, these charges give the correlated field strengths: $\vec{E} \,||\, \theta \vec{H}$, $\vec{E} \cdot \vec{H} > 0$, and both species contribute a positive amount to the mean value of the topological charge density, so that $\overline{P}_1(\theta) > 0$ and grows monotonically with θ in this interval (following increasing electric charge, $\theta/2\pi$, of the dyon). [8]

On the other side, at $\pi < \theta < 2\pi$, i.e. in the condensate of the $d_2^\theta = (1, -1 + \theta/2\pi)$ - dyons and $\bar{d}_2^\theta = (-1, 1 - \theta/2\pi)$ - antidyons, the product of signs of the magnetic and electric charges is negative for both d_2^θ - dyons and $\bar{d}_2^\theta$ - antidyons. Thus, both species contribute a negative amount to $\overline{P}_2(\theta)$, such that: $\overline{P}_2(\theta) = -\overline{P}_1(2\pi - \theta)$, and $\overline{P}(\theta)$ jumps reversing its sign at $\theta = \pi$ due to a rechargement.

Clearly, at $0 \leq \theta < \pi$, the condensate made of only the $d_1^\theta = (1,\, \theta/2\pi)$ - dyons (recalling also for a possible charged gluon pair production) can screen the same type $d_k^\theta = [\, const\, (1,\, \theta/2\pi) + (0,\, k)\,]$ - test dyon only ($k = 0, \pm 1, \pm 2, \ldots$; and the same for the d_2^θ - dyons at $\pi < \theta \leq 2\pi$). So, the heavy quark-antiquark pair will be confined at $\theta \neq \pi$.

New nontrivial phenomena arise at $\theta = \pi$. Because there are two degenerate states, i.e. the condensates of $(1, \pm\frac{1}{2})$ - dyons (and antidyons), a "mixed phase" configuration becomes possible with, for instance, each condensate filling a half of space only, and with the domain wall interpolating between them. This domain wall represents "a smeared rechargement", i.e. a smeared over space interpolation of electrically charged degrees of freedom between their corresponding vacuums, resulting in a smooth variation of the averaged densities of both type dyons through the domain wall. Surprisingly, there is no confinement inside the bulk of such domain wall.

The reason is as follows. Let us take the domain wall interpolating along the z-axis, so that at $z \to -\infty$ there is a large coherent density of d_1-dyons, and at $z \to \infty$, -that of d_2-dyons. As we move from the far left to the right, the density of d_1-dyons decreases and there appears also a small but increasing (incoherent) density of d_2-dyons. This small amount of d_2-dyons is

[8] The case $N_c = 2$ may be an exception, see footnote 5 and eq.(16), and also below.

"harmless", in the sense that its presence does not result in the screening of the corresponding charge. The reason is clear: the large coherent density of d_1-dyons keeps the d_2-dyons on the confinement, so that they can not move freely and appear only in the form of the rare and tightly connected neutral pairs, $\bar{d}_2 d_2$, fluctuating independently of each other. As we further move to the right, the d_2-dyons move more and more freely and their density increases, although they are still on the confinement. Finally, at some distance from the centre of the wall the "percolation" takes place, i.e. the d_2-dyons form a continuous coherent network and become released, so that the individual d_2-dyon can travel to arbitrary large distances (within its network). It should be emphasized that in this percolated region the coherent network of d_1-dyons still survives, so that these two networks coexist in the space.

This system shares some features in common with the mixed state of the type-II superconductor in the external magnetic field. The crucial difference is that the magnetic flux is sourceless inside the superconductor, while in the above described system there are real charges (and anticharges) inside each network. So, polarizing itself appropriately, this system of charges is capable to screen any test charge.

As we move further to the right, the density of d_2-dyons continue to increase while those of d_1 continue to decrease. Finally, at the symmetric distance to the right of the wall centre the "inverse percolation" takes place, i.e. the coherent network of d_1-dyons decays into separate independently fluctuating droplets whose average density (and size) continue to decrease with increasing z. Clearly, the picture on the right side repeats in a symmetric way those on the left one, with the d_1 and d_2 dyons interchanging their roles.

Let us consider now the heavy test quark put inside the bulk of the domain wall, i.e. inside the "percolated" region. This region obeys the properties of the "double Higgs phase". Indeed, because the charges of two dyons, $(1,\ 1/2)$ and $(1,\ -1/2)$, are linearly independent, polarizing itself appropriately this mixture of the dyon condensate networks will screen any external charge put inside, and the quark one in particular.

Finally, if the test quark is put far from the bulk of the wall, the string will originate from this point making its way toward the wall, and will be screened inside the double Higgs region.

However, if in the $\theta \neq \pi$ - vacuum the finite size ball surrounding a quark and consisting of any mixture of dyons and antidyons of any possible kind is excited, it will be unable to screen the quark charge as there is no border at infinity where the residual polarization charge will be pushed out.

We have to make a reservation about the above described picture. As was pointed out above (see footnote 5 and eq.(16)), there are reasons to expect

that the point $\theta = \pi$ is very especial for $SU(N_c = 2)$, because the vacuum energy density is likely to be exactly zero here. In this case, it is natural if the dyon condensate also approaches zero therein. The theory is expected then to have massless dyons, etc. Clearly, there will be no confinement in this phase.

4. Let us point out now that the assumption about the confinement property (at $\theta \neq \pi$) of the $SU(N_c)$ YM theory is not a pure guess, as the above discussed nonanalytical (i.e. glued) structure of the vacuum energy density, $\overline{E}_{vac}(\theta)$, is a clear signal about a phase transition at some finite temperature. Indeed, at high temperatures the θ - dependence of the free energy density is under control and is: $\sim T^4(\Lambda/T)^{N_c b_o}\cos(\theta)$, due to a rare gas of instantons. It is important for us here that it is perfectly analytic in θ, and that this $\sim \cos(\theta)$ - behaviour is T - independent, i.e. it persists when the temperature decreases. On the opposite side, at $T = 0$, the θ - dependence is nonanalytic and, clearly, this nonanalyticity survives at small temperatures. So, there should be a phase transition (confinement - deconfinement) at some critical temperature, $T_c \simeq \Lambda$, where the θ - dependence changes qualitatively.

Finally, let us show that, supposing that the monopoles indeed condense, we can find out the form of the monopole field potential. So ($N_c = 2$ and $\theta = 0$ for simplicity), let us return to the original partition function, eq.(1), and let us suppose that we have integrated out the gluon fields with two fields fixed: this time the field S and the monopole fields M and $\overline{M}$. The potential will have the form:

$$U(S, M, \overline{M}) = b_o S \ln\left(\frac{S}{\Lambda^4}\right) - Sf\left(z = \frac{\overline{M}M}{S^{1/2}}\right) , \tag{19}$$

and the stationary point equation is: $f'(z_o) = 0$. By the above assumption, the function $f(z)$ is such that this equation has a nontrivial solution: $(\overline{M}M) = z_o S^{1/2}$, i.e. with $z_o \neq 0$. Substituting it now back to eq.(19), we obtain the monopole field potential:

$$U(M, \overline{M}) = \frac{2\, b_o}{z_o^2}\left(\overline{M}M\right)^2 \ln\left(\frac{\overline{M}M}{\Lambda_M^2}\right), \quad \Lambda_M^2 = \Lambda^2 z_o e^{-f(z_o)/2b_o}. \tag{20}$$

We see that the assumption made is selfconsistent, i.e. the monopole field indeed condenses. It is sufficient to supply this potential with the simplest kinetic terms of the monopole and the (dual) neutral gluon fields, to obtain the explicit solution for the electric string.

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Appendix A

The purpose of this appendix is to present in the explicit form the asymptotic behaviour, at $\theta \to \pm i\infty$, of the vacuum energy density of the CP^N-model (the leading term at $N \gg 1$).

The partition function can be written (in Euclidean space) in the form:

$$Z = \int dn d\bar{n} dA_\mu d\lambda \, \exp\left(-\int dx I(x)\right),$$

$$I = \left\{ \overline{D_\mu n} \, D_\mu n - U \bar{n} n + \frac{N}{f_o} U - \frac{\tilde{\theta}}{2\pi} F \right\},$$

where:

$$D_\mu = \partial_\mu + i A_\mu, \; Q = \frac{F}{2\pi} = \frac{1}{2\pi} \epsilon_{\mu\nu} \partial_\mu A_\nu, \; \frac{-1}{N} T_{\mu\mu} = \frac{1}{2\pi} U = \frac{f_o}{2\pi N} \overline{D_\mu n} \, D_\mu n,$$

$U = -i\lambda$, $\tilde{\theta} = i\theta$, and f_o is the bare coupling: $f_o^{-1} = (b_o/4\pi) \ln(M_o^2/\Lambda^2)$, $b_o = 1$. Integrating out the n-field, one obtains the action:

$$\frac{1}{N} I = Tr \ln\left(-D_\mu^2 - U\right) + \frac{U}{4\pi} \ln\left(\frac{M_o^2}{\Lambda^2}\right) - \frac{\tilde{\theta}}{N} \frac{F}{2\pi}.$$

As we need the potential only, the fields U and F, which are direct analogs of the S and P fields in the YM theory, can be considered as constant ones. The determinant for this case was calculated by F. Riva [7]:

$$Tr \ln\left(-D_\mu^2 - U\right) = \frac{-1}{4\pi} \int_{1/M_o^2}^{\infty} \frac{dt}{t} \left[\frac{F}{\mathrm{sh}(Ft)} \exp\{Ut\} - \frac{1}{t} \right].$$

The stationary point equations are:

$$\ln\left(\frac{M_o^2}{\Lambda^2}\right) = \int_{1/M_o^2}^{\infty} dt \, \exp\{\overline{U} t\} \, \frac{\overline{F}}{\mathrm{sh}(\overline{F} t)}, \qquad (a1)$$

$$\int_0^\infty \frac{dz}{z\,\mathrm{sh}(z)} \exp\{\frac{\overline{U}}{\overline{F}} z\}\,[z\,\mathrm{cth}(z) - 1] = 2\,\frac{\tilde{\theta}}{N}\,. \qquad (a2)$$

We obtain from (a2) at $\tilde{\theta} \to \infty$:

$$\frac{\overline{U}(\tilde{\theta})}{\overline{F}(\tilde{\theta})} = 1 - \Delta(\tilde{\theta}), \quad \Delta(\tilde{\theta}) = \frac{N}{\tilde{\theta}} \left[1 - \frac{N}{\tilde{\theta}} \ln \frac{\tilde{\theta}}{N} + O(\frac{N}{\tilde{\theta}}) \right] ,$$

and from (a1):

$$\ln\left(\frac{M_o^2}{\Lambda^2}\right) = \ln\left(\frac{M_o^2}{\overline{F}(\tilde{\theta})}\right) + \frac{2}{\Delta(\tilde{\theta})} + O(1); \;\; \overline{F}(\tilde{\theta}) \to const \left[\frac{\tilde{\theta}}{N} \exp\left\{\frac{\tilde{\theta}}{N}\right\}\right]^{d/b_o} ,$$

where d=2 is the space-time dimension, and $b_o = 1$.

So, the behaviour of the vacuum energy density, $(1/2)T_{\mu\mu} = -N\overline{U}(\tilde{\theta})/4\pi$, is nontrivial. $\overline{U}(\tilde{\theta})$ starts with the negative value $\sim (-\Lambda^2)$ at $\tilde{\theta} = 0$, then increases monotonically with increasing $\tilde{\theta}$, passes zero at some $\tilde{\theta}_o$ and becomes positive and large, approaching $\overline{F}(\tilde{\theta})$ from below at large $\tilde{\theta}$.

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