

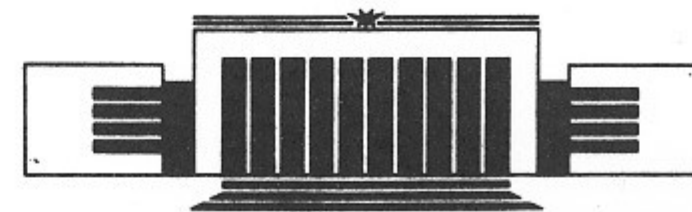


46
ИНСТИТУТ ЯДЕРНОЙ ФИЗИКИ
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INSTANTON-ANTIINSTANTON PAIR
INDUCED ASYMPTOTICS
OF PERTURBATION THEORY IN QCD

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НОВОСИБИРСК

Instanton–Antiinstanton pair induced Asymptotics of Perturbation Theory in QCD

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Abstract

The instanton–antiinstanton pair induced asymptotics of perturbation theory expansion for QCD correlators is considered. It is argued that though the true asymptotics is dominated by renormalon, the instanton-induced contribution may dominate in the intermediate asymptotics $n = 5 \div 15$.

Obtained asymptotic formulae are valid for $N_f \leq N_c$. For $N_f = N_c$ the finite nonperturbative expression for instanton–antiinstanton contribution was also found.

At $q^2 < 0$ the imaginary part of correlators in the case $N_f = N_c$ is suppressed like $1/\log(q/\Lambda)$, but the present accuracy of instanton calculations allows to fix it unambiguously.

The series of corrections to the instanton induced asymptotics of the order of $\sim (\log(n)/n)^k$ is found.

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1 Introduction. Renormalon–Instanton

The aim of this note is to try to show, what the important role may play the instanton for large order terms of perturbation theory in QCD. During last few years a lot of papers appears (see e.g. [1, 2, 3, 4, 5, 6, 7]) considering the asymptotic behaviour of perturbation theory series in QCD and QED. However the main attention was paid to the so called renormalon asymptotics. Two renormalons were considered, the ultraviolet and infrared. The usual form of the ultraviolet renormalon contribution to, for example, $R_{e^+e^- \rightarrow \text{hadrons}}$ perturbative expansion is

$$a_n \sim \left(-b \frac{\alpha_s}{4\pi}\right)^n n! , \quad (1)$$

where $b = 11/3N_c - 2/3N_f \approx 10$. In QCD the series (1) is sign alternating and at least the Borel sum of the series is well defined. The problems with summation of ultraviolet renormalon appears in QED, where all terms of series have the same sign.

Another kind of asymptotics is the infrared renormalon:

$$a_n \sim \left(\frac{b}{2} \frac{\alpha_s}{4\pi}\right)^n n! . \quad (2)$$

At large n this series turns to be much smaller than (1). The keen interest in infrared renormalon was caused by the fact that nobody knows how to

sum the series (2). The Borel sum of this series is ill defined (up to arbitrary $\sim 1/q^4$ correction).

Both ultraviolet and infrared renormalons are associated with a certain chains of Feynman graphs. However since the works of L.N.Lipatov [8] another approach is developed for large order perturbation theory estimates. In this approach one has nothing to do with Feynman graphs, but tries to find the specific ("classical") large fluctuations in the functional space making the main contribution to the high order terms of perturbative expansion. The natural example of such important fluctuations in QCD is the instanton-antiinstanton pair, but up to now only in the paper of I.I.Balitsky [9] the instanton asymptotics for $R_{e^+e^- \rightarrow \text{hadrons}}$ was considered. In present work we try to clarify further the role of instanton-antiinstanton pair effects for QCD correlators, especially in the interesting case $N_f = N_c$ (the author of [9] at $N_f = N_c$ have not found the perturbative asymptotics and use the ambiguous regularization prescription to find nonperturbative corrections).

The generic form of instanton-induced asymptotics appears to be:

$$a_{nIA} \sim \left(\frac{\alpha_s}{4\pi}\right)^n (n + 4N_c)! \quad (3)$$

The overall numerical factor in a_{nIA} may also be sufficiently large. It is seen that though the renormalons (1,2) do dominate at very large n ($n > 15$), the instanton-induced contribution may dominate in the intermediate asymptotics $n = 5 \div 15$. If so, the pure renormalon behaviour (1) will hardly be observed in directly calculated terms of perturbation theory due to a strong competition with the instanton contribution.

Of course the exactly known 3-4 terms of perturbation theory series (for β -function of QCD or $R_{e^+e^- \rightarrow \text{hadrons}}$) are much smaller than the estimate (3) and the question at what number n the perturbative series could reach the full strength (3) is open now. Moreover the corrections which are formally $\sim 1/n$ at large n may also change the value of a_n in many orders at small n . For example

$$(n + 4N_c)! = n^{4N_c} n! (1 + O(1/n)) \quad (4)$$

In this paper we have also found (and summed up) the subseries of corrections to a_{nIA} (3), which behave like $(\log(n)/n)^k$ (see (24)).

Like for the infrared renormalon (2) all terms of the series (3) have the same sign. Nevertheless the problem of summation the series (3) seems not so hopeless as the summation of renormalon. Following G.'t Hooft [10] the author of [9] proposed to rewrite the integral over the instanton-antiinstanton pair in the Borel form by considering the action as a collective variable. The

well-separated instanton-antiinstanton pair is responsible for the singular part of Borel function, while the ambiguous strongly interacting instanton and antiinstanton contribute to its smooth part. On the other hand the best way to describe the smooth part of the Borel transform is to calculate exactly the few first terms of perturbative expansion. The divergent singular part of the Borel integral corresponds to almost non-interacting pseudoparticles. The accurate subtraction from the singularity of dilute gas contribution in the toy model (double well oscillator) allowed to find the finite nonperturbative instanton-antiinstanton contribution [11]. In QCD at $N_f = N_c$ the Borel integral diverges only logarithmically and the total nonperturbative contribution from instanton-antiinstanton pair may be found by cutting the instanton size at $\rho \ll 1/\Lambda$.

At $N_f = N_c$ the imaginary part of correlators at negative q^2 cancels in the one loop approximation. Nevertheless the invariance of the instanton contribution under the renormalization group transformations allows to find the imaginary parts.

2 The ansatz

The most popular puzzle for applying the high order estimates is the calculation of $R_{e^+e^- \rightarrow \text{hadrons}}$, which is connected with the Euclidean correlator of two electromagnetic currents

$$\Pi_{\mu\nu} = \int e^{iqx} d^4x \langle j_\mu(x) j_\nu(0) \rangle, \quad (5)$$

where $j_\mu = \sum_{\text{flavours}} e_f \Psi_f^\dagger \gamma_\mu \Psi_f$. Calculation of instanton-induced contribution to (5) requires considerable algebraic efforts (e.g. the fermionic Green function in the pseudoparticle background must be used). Therefore for the sake of simplicity we will examine the correlator of two scalar currents

$$j(x) = \frac{3\alpha_s}{4\pi} [G_{\mu\nu}^a(x)]^2 \quad (6)$$

(the notations of [12] are used). This correlator, which may be useful for the glueball physics, is simple enough so the reader can check almost all steps of the calculation. Moreover the correlator of two currents (6) reproduces all the interesting features of the correlator of electromagnetic currents.

As we have said above, the strongly interacting instanton and antiinstanton correspond to a smooth part of the Borel function. It was shown by Balitsky [9] that the instanton-antiinstanton configuration relevant for the

large orders of perturbation theory is a small instanton inside the very large antiinstanton (or vice versa). The size of small instanton is regulated by the internal momentum of correlator (5) $q\rho_I \sim 1$. The size of antiinstanton (as well as the distance between the pseudoparticles $\rho_A \sim R$) becomes parametrically large as we consider the higher terms of perturbation theory.

Now let us specify the ansatz for the gauge fields. We are interesting in the instanton-antiinstanton interaction in the leading nontrivial approximation. Therefore the simple sum of instanton and antiinstanton may be used:

$$A_\mu = U_A A_\mu^A U_A^\dagger + U_I A_\mu^I U_I^\dagger, \quad (7)$$

where U_A, U_I are the constant $SU(N)$ matrices. For small instanton the singular gauge seems to be preferable

$$A_\mu^I = \frac{\bar{\eta}_{\mu\nu}(x-x_I)_\nu \rho_I^2}{(x-x_I)^2((x-x_I)^2 + \rho_I^2)}, \quad \bar{\eta}_{\mu\nu} \equiv \tau^a \bar{\eta}_{\mu\nu}^a. \quad (8)$$

Before we add the antiinstanton to (8) the singularity at $x = x_I$ is pure gauge singularity. Therefore in order to remove the singularities from all the physical quantities one may choose A_μ^A in any regular gauge which satisfy the equality $A_\mu^A(x = x_I) = 0$. For example one can rotate the BPST antiinstanton

$$A_\mu^A = S \left[\frac{\bar{\eta}_{\mu\nu}(x-x_A)_\nu}{(x-x_A)^2 + \rho_A^2} \right] S^\dagger + iS\partial_\mu S^\dagger, \quad (9)$$

$$R_\mu = (x_A - x_I)_\mu, \quad S = \exp \left\{ i \frac{\bar{\eta}_{\mu\nu} R_\nu}{R^2 + \rho_A^2} (x - x_I)_\mu \right\}.$$

It is easy to show that any other smooth matrix function $S(x)$, which allows to cancel the antiinstanton field at $x = x_I$ will lead to the same correlator.

After direct calculation the classical action of the instanton-antiinstanton configuration may be found with the usual dipole-dipole interaction of pseudoparticles

$$S_{IA} = \frac{4\pi}{\alpha_s} \{1 - \xi h\}, \quad \xi = \frac{\rho_I^2 \rho_A^2}{(R^2 + \rho_A^2)^2}, \quad h = 2|\text{Tr}O|^2 - \text{Tr}OO^\dagger, \quad (10)$$

and O is the upper left 2×2 corner of the matrix $U = U_A^\dagger U_I$ (7).

Another part of the problem, extremely sensitive to the instanton-antiinstanton interaction, is the fermionic determinant. It may be shown, that for each

flavor of massless fermions the two anomalously small eigenvalues of Dirac operator \hat{D} appears

$$\lambda_{1,2} = \pm \frac{2\rho_I \rho_A}{(\rho_A^2 + R^2)^{3/2}} |\text{Tr}O|. \quad (11)$$

3 Calculation of correlator

After we have defined the gauge field configuration it is easy to write down the instanton-antiinstanton contribution to the correlator of two currents (6). Everywhere it is possible the notations of [9] are used.

$$\begin{aligned} \Pi(q) &= \int e^{iqx} d^4x \langle j(x)j(0) \rangle = \\ &= 2 \int j_I(x)j_I(0) e^{iqx} [4\xi^{3/2} |\text{Tr}O|^2]^{N_f} \times \\ &\times \exp \left\{ \frac{4\pi}{\alpha_s} \xi h \right\} \frac{d(\rho_I)}{\rho_I^5} \frac{d(\rho_A)}{\rho_A^5} dx dx_I dx_A d\rho_I d\rho_A dU, \end{aligned} \quad (12)$$

where

$$j_I(x) = \frac{36}{\pi^2} \frac{\rho^4}{((x-x_I)^2 + \rho_I^2)^4}, \quad (13)$$

and the instanton density [13]

$$d(\rho) = A \left(\frac{2\pi}{\alpha_s(\rho)} \right)^{2N_c} \exp \left(-\frac{2\pi}{\alpha_s(\rho)} \right). \quad (14)$$

The factor 2 in front of the integral in (12) accounts for the equal contribution from small antiinstanton and large instanton.

We will also use the well known two-loop formula

$$\frac{4\pi}{\alpha_s(q)} = b \log \left(\frac{q^2}{\Lambda^2} \right) + \frac{b'}{b} \log \left(\log \left(\frac{q^2}{\Lambda^2} \right) \right) + \dots \quad (15)$$

In the most interesting case $N_f = N_c = 3$ $b = 9$ and $b' = 64$.

Before passing to the formal computations let us say a few words about the existence of integral (12) as a whole. The most ambiguous part of the problem is the integration over large antiinstanton coordinates ρ_A and $R = x_A - x_I$. There are two competing effects. The factor $d(\rho_A) \sim \rho_A^b$ tends to make the integral over ρ_A divergent. On the other hand, the almost zero

fermionic modes (11) tends to suppress the large ρ_A and R contribution. If $N_f \leq N_c$ the first effect dominates and the integral (12) diverges at large ρ_A . Nevertheless just in this case the well defined instanton induced asymptotics of perturbation theory may be extracted from (12). For nonperturbative calculation of the whole integral (12) at $N_f \leq N_c$ the new physical income is necessary. Below we will show how to perform this integration for $N_f = N_c = 3$.

If the number of massless flavours is sufficiently large ($N_f > N_c$) the attraction due to fermionic zero modes prevails. As a result the approximation of almost noninteracting pseudoparticles breaks down ($\rho_A \sim R \sim \rho_I$) and the instantonic approach itself became ambiguous.

Formulae (14), (15) allow to extract the ρ_A dependent part from (12)

$$d(\rho_A) = \phi(\rho_I^2/\rho_A^2) \left(\frac{\rho_I}{\rho_A}\right)^b d(\rho_I), \quad \phi(x) = \left[1 + b \frac{\alpha_s}{4\pi} \log(x)\right]^{2N_c - \frac{b'}{2b}} \quad (16)$$

Everywhere below we suppose $\alpha_s = \alpha_s(q) \simeq \alpha_s(\rho_I)$. For calculation of leading perturbative asymptotics one may assume $\phi(x) \equiv 1$ (as it was done in [9]), but in order to calculate the nonperturbative value of correlator (12) the function $\phi(x)$ of the form (16) should be used. Moreover the corrections $\sim (\alpha_s \log(x))^k$ contained in the $\phi(x)$ leads to an important subseries of preasymptotic corrections $\sim (\log(n)/n)^k$ to the leading asymptotics of the perturbative expansion. Therefore below we shall use the function $\phi(x)$, though suppose that its argument is small enough $|\log(x)| \gg 1$.

Now we would like to integrate over ρ_A and $R = x_A - x_I$ for a fixed value of ξ

$$\int \phi(\rho_I^2/\rho_A^2) \rho_A^{b-5} \delta\left(\frac{\rho_I^2 \rho_A^2}{(R^2 + \rho_A^2)^2} - \xi\right) d\rho_A d^4 R = \frac{\pi^2}{2(b-2)(b-1)} \frac{\rho_I^b}{\xi^{b/2+1}} \phi(\xi). \quad (17)$$

The part of (12) depending on ρ_I , x and x_I in the leading approximation over α_s gives:

$$\int j_I(x) j_I(0) e^{iqx} \rho^{2b-5} d^4 x d^4 x_I d\rho = 9 \frac{2^{2b-3} \Gamma(b+2) \Gamma^2(b) \Gamma(b-2)}{q^{2b-4} \Gamma(2b)}. \quad (18)$$

After all the correlator (12) reads

$$\Pi(q) = Const q^4 d^2(1/q) \int |Tr O|^{2N_f} \exp\left\{\frac{4\pi}{\alpha_s} \xi h\right\} \frac{\phi(\xi) d\xi}{\xi^{\frac{b-3N_f}{2}+1}} dU. \quad (19)$$

The last step, which allows to rewrite the correlator in the form of Borel integral is to introduce the variable $t = 1 - \xi h$:

$$\Pi(q) = 9\pi^2 2^{2(b+N_f-2N_c)-3} \frac{(b+1)!(b-1)!(b-3)!^2}{(2b-1)!} A^2 q^4 \left(\frac{4\pi}{\alpha_s(q)}\right)^{4N_c} \quad (20)$$

$$\left\{ \int_0^1 dt \phi(1-t)(1-t)^{\frac{3N_f-b}{2}-1} e^{-\frac{4\pi}{\alpha_s} t} \langle |Tr O|^{2N_f} \Theta(h) h^{\frac{b-3N_f}{2}} \rangle + \int_1^\infty dt \phi(t-1)(t-1)^{\frac{3N_f-b}{2}-1} e^{-\frac{4\pi}{\alpha_s} t} \langle |Tr O|^{2N_f} \Theta(-h) (-h)^{\frac{b-3N_f}{2}} \rangle \right\}.$$

Here Θ -function equals 0 or 1 in accordance with sign of its argument and $\langle \dots \rangle$ means averaging over the orientation of the matrix U .

Essentially the same expression as (20) (except for the overall power of q and numerical factors and with ϕ replaced by 1) was found in [9] for the correlator of electromagnetic currents (5).

4 Analyzing the result

The first conclusion which is to be done is that the result (20) may be used only for $N_f \leq N_c$ because the integral over orientations of the matrix U diverges at $h = 0$ if $N_f > N_c$. This means that our method can not be applied without strong modification, for example, to calculation of $\Gamma_{Z_0 \rightarrow hadrons}$ ($N_c = 3, N_f = 5$).

Another interesting application for asymptotic formulae will be the case $N_f = N_c = 3$. The averaging over U for $SU(3)$ group gives

$$\langle |Tr O|^6 \rangle = \frac{7}{5},$$

$$\langle |Tr O|^6 \Theta(h) \rangle = 1.37; \quad \langle |Tr O|^6 \Theta(-h) \rangle = 0.03. \quad (21)$$

Here the averages with Θ -function are the numerical estimates. The accuracy of the last (small) value is expected to be not worse than 20%.

Thus the final expression for instanton-antiinstanton pair contribution at $N_f = N_c = 3$ reads

$$\Pi(q) = \frac{9}{\pi^2} e^{5/3} \frac{10!7!6!^2}{17!} q^4 \left(\frac{4\pi}{\alpha_s}\right)^{12} \quad (22)$$

$$\left\{ 1.37 \int_0^1 \phi(1-t) \frac{\exp(-\frac{4\pi}{\alpha_s} t)}{1-t} dt + 0.03 \int_1^\infty \phi(t-1) \frac{\exp(-\frac{4\pi}{\alpha_s} t)}{t-1} dt \right\}.$$

This expression is enough to find the leading asymptotics of perturbative expansion for $\Pi(q)$:

$$\Pi(q) = q^4 \sum \Pi_n \left(\frac{4\pi}{\alpha_s}\right)^n, \quad \Pi_n = \frac{9}{\pi^2} e^{5/3} \frac{10!7!6!^2}{17!} 1.37(n+12)! \quad (23)$$

We see that instanton-antiinstanton induced contribution to the series of perturbation theory at $n \sim 10$ do has a huge enhancement $(n+12)!$ compared with $b^n n!$ for renormalon. The complete calculation of the $\sim 1/n$ corrections to the leading asymptotics requires a considerable efforts even in the simple toy model [11]. Nevertheless one may try to find any particular corrections which are enhanced in some way. The set of such enhanced $\sim \log(n)/n$ corrections appears from the expansion of $\phi(1-t)$ in (22) in powers of α_s . Let us remind, that physically with $\phi(x)$ one takes into account the running of the coupling $\alpha_s(\rho_A)$ which describes the large antiinstanton. It seems very unlikely if one can find any other effect which lead to such a large corrections $\sim (\log(n)/n)^k$. Under this assumption we find:

$$\Pi_n = 176 \left[1 - 9 \frac{\log(n)}{n}\right]^{22/9} (n+12)! \quad (24)$$

Of course this result may be used only if $n \gg \log(n)$. Though the expression (22) provides us with the asymptotic of perturbative series, both integrals in (22) diverge at $t = 1$. In the configuration space these divergences are related to the integration over almost noninteracting instanton and antiinstanton. Because the divergence is only logarithmic one can try to use the physical intuition in order to restrict the range of integration in (22). Anyway the natural cut for ρ_A seems to be $\rho_A \ll 1/\Lambda_{QCD}$, or in terms of t

$$|t-1|_{min} \sim \left(\frac{\rho_I}{\rho_{Amax}}\right)^2 \ll \frac{\Lambda^2}{q^2} \quad (25)$$

If so the nonperturbative part of (22) may be found explicitly (up to corrections $\sim \alpha_s$).

$$\Pi(q) = 129q^4 \left(\frac{4\pi}{\alpha_s}\right)^{12} \left\{ 1.37P \int_0^\infty \phi(|1-t|) \frac{\exp(-\frac{4\pi}{\alpha_s}t)}{1-t} dt + \frac{7}{155} \left(\frac{4\pi}{\alpha_s}\right) \exp(-\frac{4\pi}{\alpha_s}) \right\} \quad (26)$$

Here P means the principal value integral. Let us stress that if one replaces $\phi(x)$ by 1 in (22), the nonperturbative part of (26) would be 31/9 times

larger. Effectively the integration over t in (22),(26) may be thought as the integration over the size of large antiinstanton. The size of antiinstanton also may be determined through the coupling constant $\alpha_s(\rho_A)$ (the logarithmic scale $\alpha_s(\rho_A)^{-1} \sim -\log(\rho_A \Lambda)$). The remarkable feature of our result is that all the values of $\alpha_s(\rho_A)$ contribute to the nonperturbative part (26) in the whole range $\alpha_s(\rho_I) < \alpha_s(\rho_A) \ll 1$.

5 Imaginary part

In order to find the physical quantities such as the inclusive widths and cross-sections one have to consider the imaginary part of correlators, which appears after the analytic continuation to Minkovsky momentum $Im(\Pi(-q^2 + i\varepsilon))$. In the lowest order in α_s both the singular part and the nonperturbative corrections in (26) behave like $(\Lambda/q)^{2b}$. Thus for $N_f = N_c = 3$ ($b = 9$) the imaginary part cancels. In this case the second term of the expansion of α_s should be considered (15) and the analytic continuation of (26) gives:

$$\begin{aligned} \frac{1}{q^4} Im(\Pi(-q^2 + i\varepsilon)) &= 2.4 \cdot 10^4 \left(\frac{4\pi}{\alpha_s}\right)^{11} P \times \quad (27) \\ &\times \int_0^\infty \psi(|1-t|) \frac{\exp(-\frac{4\pi}{\alpha_s}t)}{1-t} dt + 0.9 \cdot 10^3 \left(\frac{4\pi}{\alpha_s}\right)^{12} \exp(-\frac{4\pi}{\alpha_s}), \\ \psi(x) &= \left(1 + \frac{43}{6} \frac{4\pi}{\alpha_s} \log(x)\right) \left(1 + 9 \frac{4\pi}{\alpha_s} \log(x)\right)^{13/9} \end{aligned}$$

Very similar expression may be found for $R_{e^+e^- \rightarrow hadrons}$. Although the considerable additional efforts are necessary for this calculation.

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