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INSTABILITY OF WEAKLY NONLINEAR CHAOTIC STRUCTURES

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НОВОСИБИРСК

Instability of weakly nonlinear chaotic structures

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ABSTRACT

A linear oscillator driven by periodic perturbation is considered. The infinite connected chaotic structures in phase plane emerge when the perturbation is of the form of the periodic δ-function and the exact resonance condition is fulfilled [1]. These structures are shown to be unstable and completely destroyed if the duration of the perturbation kick is arbitrarily short but finite.

A linear oscillator driven by periodic nonlinear perturbation is often used as a dynamical model for some physical problems. Its motion is described by the Hamiltonian:

$$H(x,p,t) = \frac{p^2 + \omega_0^2 x^2}{2} + \varepsilon V(x,p,t), \tag{1}$$

where the perturbation $V(x,p,t+T) = V(x,p,t)$ is a time–periodic function. In Eq.(1) the nonlinearity depends on the weak (as $\varepsilon \to 0$) perturbation only and, if the resonance occurs, we have the weakly nonlinear resonance system, on which the extension of the KAM theory is impossible. Dynamics of the systems, which may be very unexpected, was extensively studied (see, e.g., [1]).

Let perturbation in Eq.(1) contain only one harmonic. Then

$$H(x,p,t) = \frac{p^2 + \omega_0^2 x^2}{2} + \varepsilon \cos(x - \Omega t). \tag{2}$$

If we put $\omega_0 = 0$ the model describes a single strongly nonlinear resonance (SNR) which is completely integrable with no trace of a chaos [2]. Yet, for any $\omega_0 \neq 0$ nonlinearity becomes weak (as $\varepsilon \to 0$) and the motion drastically changes. If Ω/ω_0 is an integer, the model describes a single weakly nonlinear resonance (WNR) which has a very complicated chaotic component [3].

In present paper we are going to discuss in detail another case of Eq.(1) with infinite number of harmonics:

$$H(x,p,t) = \frac{p^2 + \omega_0^2 x^2}{2} + \varepsilon \cos x \, \delta_T(t), \tag{3}$$

where $\delta_T(t)$ is the T–periodic δ–function and the perturbation parameter $\varepsilon << 1$ is small. The model (3) may represent the motion of a charged particle in both a magnetic field (Larmor's frequency ω_0) and the field of a perpendicularly propagating wave packet [1]. If we put $\omega_0 = 0$ the model describes a strongly nonlinear system with the infinite set of interacting resonances and their chaotic layers. For a sufficiently small perturbation $\varepsilon << 1$ the layers of different resonances are separated from each other by stable invariant tori and an unbounded motion is impossible [4]. But for any $\omega_0 \neq 0$ and $T = 2\pi/n\omega_0$ with any integer n an infinite and uniform connected chaotic web emerges on phase plane [1]. The unbounded motion of a particle along this web is possible.

The web is unstable under detuning from the exact resonance condition ($2\pi/T - n\omega_0 \neq 0$), as was shown in [5] (see also [1]). We study another kind of the instability due to a finite kick's width.

Let us replace δ–function in Eq.(3) by another one $F(t)$, which has a form of periodic rectangular function of length Δ and height $1/\Delta$ located in the middle of every period T. Then

$$H_n(x,p,t) = \frac{p^2 + \omega_0^2 x^2}{2} + \varepsilon \cos x \, F(t) = \omega_0 I + \varepsilon \cos(\rho \cos \theta) \, F(t) , \qquad (4)$$

where

$$F(t) = \frac{a_0}{2} + \sum_{k \geq 1} a_k \cos(kn\omega_0 t),$$

$$a_k = \frac{n\omega_0}{\pi}(-1)^k \frac{\sin(kn\omega_0\Delta/2)}{(kn\omega_0\Delta/2)} ; \qquad k = 0,\, 1,\, 2, \ldots .$$

In Eq.(4) $x = \rho \cos(\theta)$, $p = -\rho\omega_0 \sin(\theta)$ and $\rho = (2I/\omega_0)^{1/2}$ is the amplitude of the unperturbed oscillations. The exact resonance condition $T = 2\pi/n\omega_0$ with some integer n is supposed to be fulfilled. This condition is the only one and the model describes a single WNR (compare with Eq.(2)).

Introducing a new slow phase $\varphi = \theta - \omega_0 t$, new time $\tau = \varepsilon n \omega_0 t / 2\pi$, expanding the perturbation in the Bessel functions and averaging over the fast oscillation we have arrive at the first-order resonance Hamiltonian (for more details see [1,2]):

$$\widetilde{H}_{\Delta,n}(I,\varphi,\Delta) = J_0(\rho) + 2 \sum_{k \geq 1} (-1)^{k+\frac{kn}{2}} J_{kn}(\rho) \frac{\sin(kn\omega_0\Delta/2)}{(kn\omega_0\Delta/2)} \cos(k\,n\,\varphi) \quad (5)$$

where kn are even. For $\Delta = 0$ the perturbation has a form of periodic δ-function as in Eq.(3) and we will use for this case a special symbol $\widetilde{H}_{\delta,n}$:

$$\widetilde{H}_{\delta,n}(I,\varphi) = \widetilde{H}_{\Delta,n}(I,\varphi,\Delta = 0) = J_0(\rho) + 2 \sum_{k \geq 1} (-1)^{k+\frac{kn}{2}} J_{kn}(\rho) \cos(k\,n\,\varphi)$$

with kn even. (6)

Expanding $\sin(kn\omega_0\Delta/2)/(kn\omega_0\Delta/2)$ and regrouping terms in Eq.(5) we obtain:

$$\widetilde{H}_{\Delta,n}(I,\varphi,\Delta) = \widetilde{H}_{\delta,n}(I,\varphi) + \sum_{m \geq 1} \frac{A_{2m}(I,\varphi)}{(2m+1)!} \left(\frac{\omega_0\Delta}{2}\right)^{2m} , \qquad (7)$$

$$A_{2m}(I,\varphi) = 2 \sum_{k \geq 1} (-1)^{\frac{kn}{2}+k+m} J_{kn}(\rho)(kn)^{2m} \cos(k\,n\,\varphi) ,$$

where kn are even. Using $2m$-fold integration of $A_{2m}(I,\varphi)$ with respect to φ and representing the results in terms of $\widetilde{H}_{\delta,n}(I,\varphi)$ and its derivatives we obtain an interesting relation between the two Hamiltonians:

$$\widetilde{H}_{\Delta,n}(I,\varphi,\Delta) = \sum_{m \geq 0} \frac{(\omega_0\Delta/2)^{2m}}{(2m+1)!} \frac{\partial^{2m}}{\partial\varphi^{2m}} \widetilde{H}_{\delta,n}(I,\varphi) . \qquad (8)$$

A part of the infinite and uniform web for the resonance $n = 4$ and $F(t) = \delta_T(t)$ is reproduced in Fig.1. This so-called 'Kicked Harper Model' (KHM) is extensively studied now by many researchers [6].

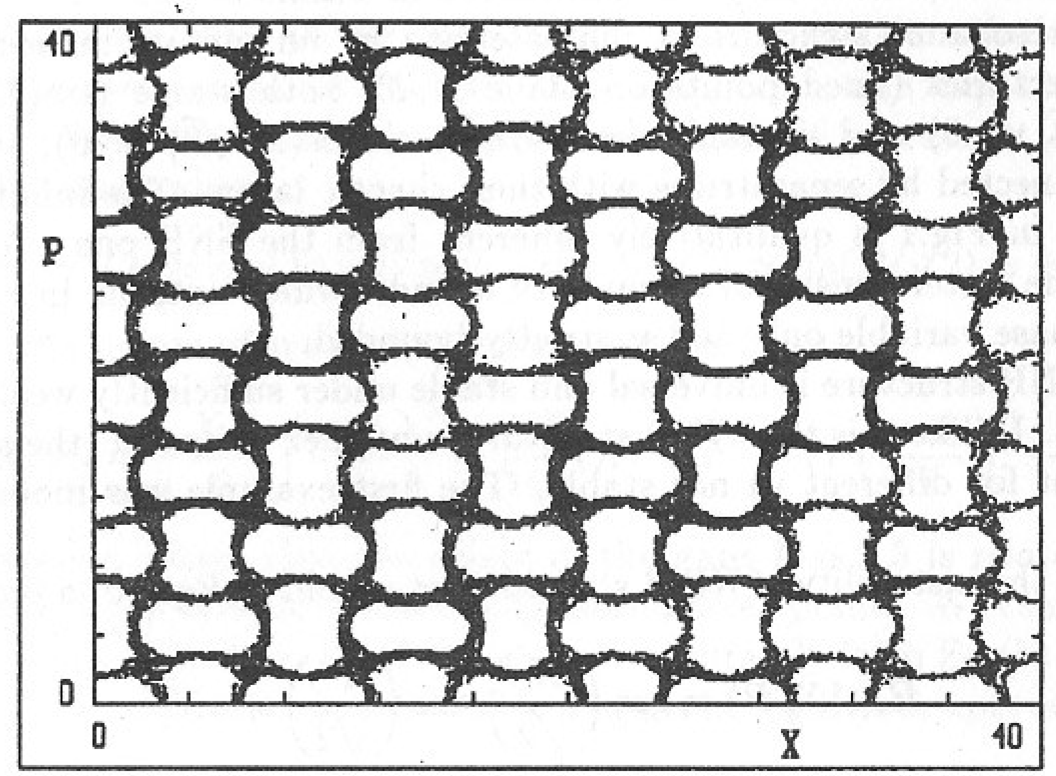

Fig. 1. Computer simulation of model Eq.(4) with $n = 4$; $\omega_0 = 1$; $\varepsilon = \pi/2$. The perturbation is periodic δ-function. X, P - are slow variables (9) shown at $t = 0 \bmod T$ with initial values in the vicinity of the saddle point $X = P = \pi/\sqrt{2}$.

A compact form of the first-order resonance Hamiltonian (6) may be derived with the help of the following identity ([7], $m \geq 1$ is any integer):

$$J_0(\rho) + 2\sum_{k\geq 1}(-1)^{mk} J_{2mk}(\rho)\cos(2mk\vartheta) =$$

$$\frac{1}{m}\cos(\rho\cos\vartheta) + \frac{1}{m}\sum_{j=1}^{m-1}\cos\left[\rho\cos(\frac{j\pi}{m}+\vartheta)\right],$$

For KHM, by putting in the identity $m = 2$, $\vartheta = \varphi + \pi/4$ and using slow variables

$$X = \rho\cos\varphi, \quad P = -\rho\omega_0\sin\varphi \tag{9}$$

we obtain :

$$\widetilde{H}_{\delta,4}(X,P) = \cos\left(\frac{X}{\sqrt{2}}\right)\cos\left(\frac{P}{\sqrt{2}}\right), \tag{10}$$

It coincides with the result in [1] up to a turn of the Cartesian coordinate axes. Exactly the same form has the first–order resonance Hamiltonian for model Eq.(2) with the only harmonic sufficiently far from the origin $X = P = 0$. Its structure and instability was considered in details in [3].

This resonance structure is characterized by an infinite lattice of periodic trajectories (fixed points on plane X, P) both stable ($\sin(X/\sqrt{2}) \approx \sin(P/\sqrt{2}) \approx 0$) and unstable ($\cos(X/\sqrt{2}) \approx \cos(P/\sqrt{2}) \approx 0$), the latter being connected by separatrices with their chaotic layers. The infinite WNR structure in Fig.1 is qualitatively different from the SNR one. As is well known, the latter consist of a 'chain of islands' which extends in the direction of phase variable only and is strictly bounded.

The SNR structure is universal and stable under sufficiently weak perturbation [4]. Unlike this the WNR structure is neither universal (the structure is different for different n) nor stable. The first example was model Eq.(2) [3].

To see the instability of KHM structure let us add to Eq.(10) a term linear in X:

$$\widetilde{H}_{\delta,4}(X,P) = \cos\left(\frac{X}{\sqrt{2}}\right)\cos\left(\frac{P}{\sqrt{2}}\right) + aX, \tag{11}$$

Then vertical separatrices ($X = const$) all remain unchanged but horizontal ones ($P = const$) are destroyed because of the difference $\Delta\widetilde{H}_{\delta,4}$ in $\widetilde{H}_{\delta,4}$ between the two neighboring fixed points. Remarkably, an arbitrarily small perturbation ($a \to 0$) qualitatively changes the structure making all the

rows of resonance cells disconnected by narrow vertical gaps. For small $a > 0$ the width of a gap at $X = 3\pi/\sqrt{2} \bmod 2\sqrt{2}\pi$ and $P = 0 \bmod 2\sqrt{2}\pi$ is:

$$\Delta X \approx 2\left|\frac{\Delta\widetilde{H}_{\delta,4}}{\partial\widetilde{H}_{\delta,4}/\partial X}\right| \approx 2\,\pi\,|a| \tag{12}$$

The motion inside a gap is unbounded in P. One can realize such an 'accelerating regime' for KHM by adding to Hamiltonian (4) the time–dependent resonant perturbation:

$$\Delta H_4 = \frac{\pi a}{\omega_0} x\cos(\Omega t), \tag{13}$$

where $\Omega = \omega_0$ (the case of a linear resonance in nonlinear web, see [3]).

Now, let the kick in Eq.(4) be of a finite width Δ . Phase portrait of KHM for $\Delta/T = 0.2$ is given in Fig.2 (the other parameters are as in Fig.1). One may see the qualitative change in the phase plane structure: the infinite lattice of fixed points remains almost unchanged but the chaotic web is completely destroyed by the extremely complicated net of gaps.

In Fig.3 the levels of resonance Hamiltonian $\widetilde{H}_{\Delta,4}$ (5) are shown. Remarkably, the analytical construction well reproduces the most principal features of KHM phase structure Fig.2 (without chaotic components).

To clarify the origin of the gaps in Figs.2,3. we use the relation Eq.(8) between the two resonance Hamiltonians $\widetilde{H}_{\Delta,4}$, $\widetilde{H}_{\delta,4}$ and Eq.(10) for the latter. The following calculations will be simplified if we restrict ourself to the saddle points where

$$\cos(X/\sqrt{2}) \approx \cos(P/\sqrt{2}) \approx 0; \quad \sin(X/\sqrt{2}) \approx \sin(P/\sqrt{2}) \approx \pm 1\,.$$

In the saddle points of system (4) with $n = 4$ we have:

$$\widetilde{H}_{\Delta,4}(X,P) \approx -\frac{XP}{6}\left(\frac{\omega_0\Delta}{2}\right)^2 + \frac{XP}{120}(X^2+P^2+7)\left(\frac{\omega_0\Delta}{2}\right)^4 - \ldots \tag{14}$$

This expression shows that the origin of the gaps Figs.2,3 is related to the difference in $\widetilde{H}_{\Delta,4}$ between the two neighboring fixed points. We can estimate the width of the gap from Eq.(14) and more accurately from Eq.(5). For several low (nearest to the origin $X = P = 0$) gaps in Fig.3 both approximations give close results.

Dependence of the resonance Hamiltonian $\widetilde{H}_{\Delta,4}$ on X, P leads also to a change of the frequencies of small oscillations and to a decrease in the width of chaotic layers when moving away from the origin. For low gaps the chaotic layers may close the gaps completely but at some distance on the gaps remain open and block the diffusion (the first open gap is shown on Fig.2).

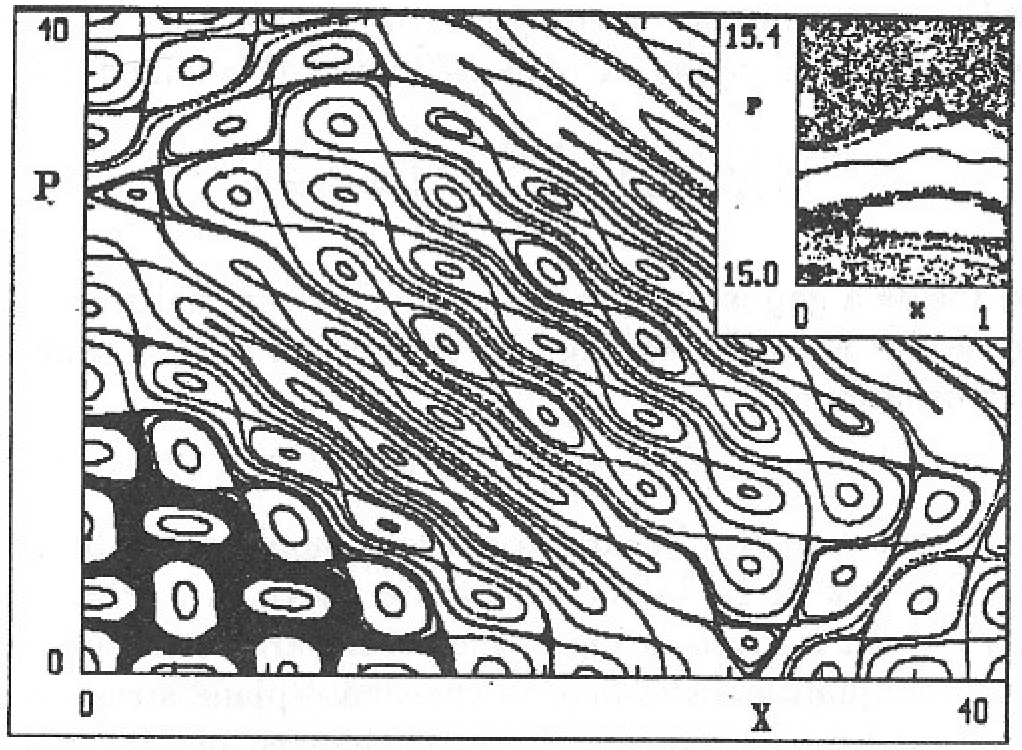

Fig. 2. The same as in Fig.1 with a finite $\Delta/T = 0.2$. Insert: enlarged part of the first open gap and an invariant trajectory inside it.

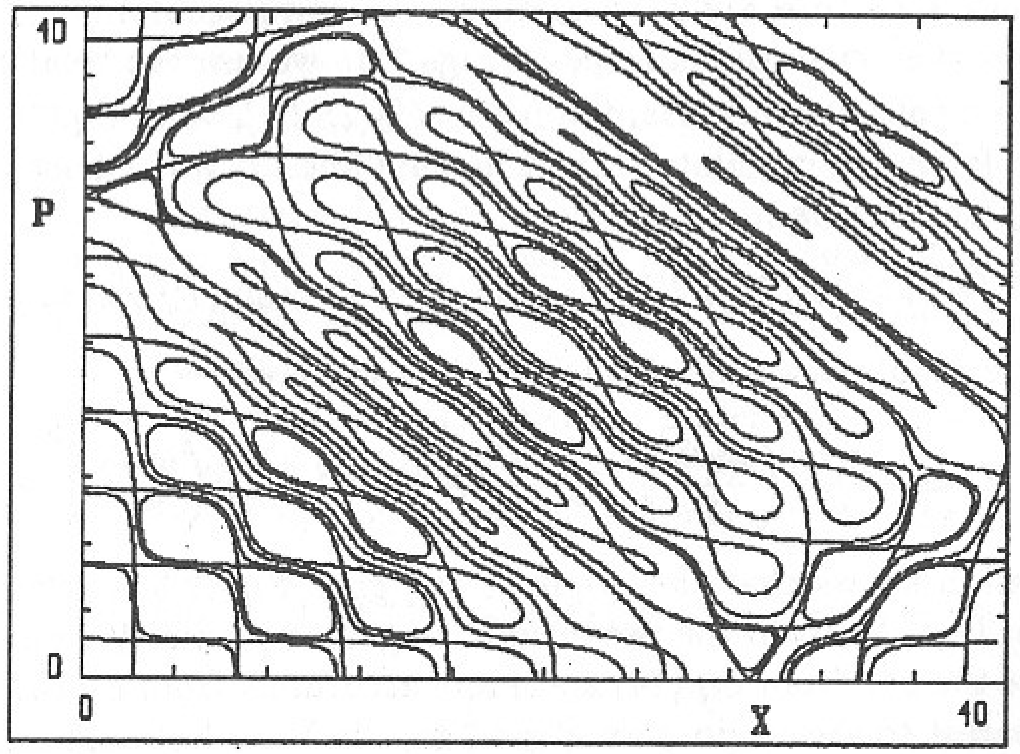

Fig. 3. The levels of resonance Hamiltonian $\widetilde{H}_{\Delta,4}$ using first 15 terms in Eq.(5). All parameters are the same as in Fig.2.

In conclusion we note, that the weakly nonlinear chaotic structure is infinite connected web only if the perturbation has the form of the periodic δ-function and the exact resonance condition is fulfilled. As was shown above, the web is not stable. Under a weak perturbation or for finite kick's width it becomes disconnected by many narrow gaps.

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