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SINGULARITIES OF FAMILIES OF EVOLVENTS IN THE NEIGHBORHOOD OF AN INFLECTION POINT OF THE CURVE, AND THE GROUP H_3 GENERATED BY REFLECTIONS

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In [1] the singularities of the family of evolvents of a plane curve in the neighborhood of a convexity point of the curve were studied. Namely, it was shown that there is a diffeomorphism of the plane which reduces this family to a unique normal form. In the present paper we solve a similar problem for inflection points of the curve.

The investigation of the singularities of the families of evolvents of a curve in Euclidean plane is called in the physical interpretation the plane problem of bypassing an obstacle. Here the curve is seen as an obstacle and the evolvents as the fronts of a light flux which moves bypassing the obstacle. Every front is the level line of a (multivalued) time-function which measures the time the light needs to reach a given point of the plane bypassing the obstacle. A light flux which envelops the obstacle (i.e., which moves along it with unit speed) yields a natural parametrization of the obstacle, $r:\mathbf{R} \to \mathbf{R}^2$. The fronts are images of the lines $s_1 + s_2 = \text{const}$ under the map $\varphi: \mathbf{R}^2 \to \mathbf{R}^2$, $\varphi(s_1, s_2) = r(s_1) + s_2\dot{r}\cdot(s_1)$. The quantity $s_1 + s_2 = t$ is the value of the time-function on the corresponding front.

In this paper we find the normal form of the graph of the time-function with respect to diffeomorphisms of the phase-space which preserve the value of the time coordinate, in the neighborhood of an inflection point of the obstacle.

By straightforward computations one may readily establish the form of the fronts in the neighborhood of an inflection point of the obstacle (such computations were first done by V. I. Arnol'd). Thus, the front passing through the inflection point has at this point a singularity 5/3 (i.e., it is locally diffeomorphic to the curve $x^5 = y^3$ at zero); the other fronts display two singularities: one (of type 3/2) on the obstacle, and a second (of type 5/2) on the line tangent to the obstacle at the inflection point (Fig. 1). The graph of the time-function in a neighborhood of the inflection point is showed in Fig. 2.

A. B. Givental' made the observation that the manifold of irregular orbits of the group H_3 generated by reflections, which was earlier studied by Lyasko [5], looks the same way, and he conjectured that these two objects are actually diffeomorphic.

The main result of our paper is a proof of this conjecture.

THEOREM 1. The graph of the time-function in the plane problem of bypassing an obstacle is diffeomorphic, in a neighborhood of an inflection point, with the manifold of irregular orbits of the group H_3.

In other words, there is a local diffeomorphism of space—time in the space of all orbits of H_3, which takes the graph of the time-function into the manifold of irregular orbits.

The orbit space of the group H_3 is a three-dimensional space with a basis provided by a basis of invariants of H_3. This basis consists of polynomials of order 2, 6, and 10 (see, for example, [7]).

THEOREM 2. The diffeomorphism of Theorem 1 can be selected to take the time coordinate (up to a constant shift) into the coordinate corresponding to the vector of degree 2 in the aforementioned basis.

In the three-dimensional problem of bypassing an obstacle, the light flux enveloping the

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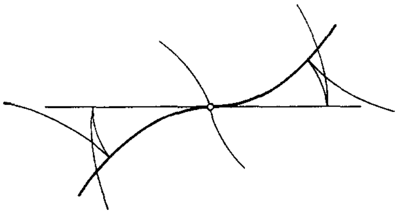

Fig. 1.

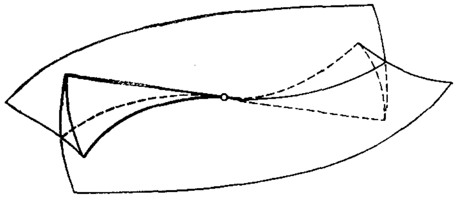

Fig. 2.

obstacle (surface) is a beam (one-parameter family) of geodesics on it. For obstacles in general position this beam has an asymptotic direction at the points of some curve lying on the obstacle.

THEOREM 3. In the three-dimensional problem of bypassing an obstacle, the front passing through a general point of the asymptotic direction of the enveloping beam is locally diffeomorphic at this point to the manifold of irreglar orbits of the group H_3.

Proof of Theorem 1. Using the Huygens principle [3], we can think of any front as a radiating surface. All the other fronts are fronts of this surface, and the obstacle itself plays the role of a caustic [3, 4]. The germ of the graph of the time-function at an inflection point P of the obstacle can be entirely reconstructed from the germ of a front Γ, which does not pass through P, at its singular point Q of type 5/2.

Consider the family of functions $f(x, y, t) = \rho(x, y) - t$, where x are coordinates in the plane in a neighborhood of the point Q, y are coordinates in the plane in a neighborhood of the point P, t denotes time and ρ is the distance function in the plane (here y and t are regarded as parameters). The restriction of this family to the front Γ is the generating family for the Legendrean mapping defined by the light flux [4]. The graph of the time-function is the set of those values of the parameters for which the zero level set of the function $f(\cdot, y, t)$ is tangent to Γ (by tangency at a singular point of the front we mean tangency in the direction of the edge of the front).

Thus, the problem is now to reduce to normal form a family of functions given in a plane with a boundary which has a singularity 5/2 [5]. We first bring the boundary to the normal form $x_1^5 = x_2^2$ by means of a change of the x-coordinate. Using the results of [5] one may readily see that our family can be put in the form $f(x, \lambda) = x_2 + \varepsilon x_1^3 + \lambda_1 x_1^2 + \lambda_2 x_1 + \lambda_3$, where $\varepsilon = \varepsilon(\lambda)$, and λ are new coordinates in the parameter plane ($\lambda = 0$ corresponds to the point under consideration on the graph of the time-function). In [5] it was proved that the above set of values of the parameters is, independent of the function $\varepsilon(\lambda)$, diffeomorphic to the manifold of irregular orbits of H_3.

Proof of Theorem 2. This theorem is a corollary of Theorem 1 and the results of [2]. To apply the latter, we need only verify that the derivative at zero in the direction of the base vector of degree 2 of the function resulting by transporting the time-function to the orbit space of H_3 via the diffeomorphism of Theorem 1 does not vanish. This follows from the fact that the time function is not singular on the cuspidal edge 3/2 of its graph, whereas the differentials of the invariants of degrees 6 and 10 vanish on this edge.

Proof of Theorem 3. As indicated in [6], at a generic point of the asymptotic line tangent to the beam, the front has a singularity of the type of a cuspidal edge 5/2 (i.e., in the coordinates $x = (x_1, x_2, x_3)$, it is locally given by the equation $x_1^5 = x_2^2$). Now define the generating family f(x, y) as in Theorem 1, fixing a value $t = t_0$ so that $f(0, 0) = 0$. Using the technique of paper [5], we put our family in the form $f(x, \lambda) = x_2 + x_3^2 + \varepsilon x_1^3 + \lambda_1 x_1^2 + \lambda_2 x + \lambda_3$. Note that since the boundary is a cylindrical surface whose generatrices are parallel to the x_3-axis, tangency of the zero level-set of the function $f(\cdot, \lambda)$ can occur only for $x_3 = 0$ (because $\partial f/\partial x_3 = 0$). Consequently, the set of bifurcational values of the parameter is the same as in Theorem 1.

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A CLASS OF SYSTEMS OF VOLTERRA INTEGRAL EQUATIONS OF FIRST KIND

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Let us denote the set of the matrices acting in E_n by M, where E_n is the n-dimensional complex (real) Euclidean space. The set M becomes a Hilbert space if the scalar product $[A, B]_M$ of its elements $A = (a_{ij})$ and $B = (b_{ij})$ is defined by the equation

$$[A, B]_M = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\bar{b}_{ij}.$$

Definition. A matrix-valued function $A(t, s) = (a_{ij}(t, s)) \in L_2([a, b] \times [a, b]; M)$ is called a Hermitian matrix-valued kernel if the following conditions are fulfilled: $a_{ij}(t, s) = \bar{a}_{ji}(s, t)$, $(i, j = 1, \ldots, n)$ for almost all $(t, s) \in [a, b] \times [a, b]$.

The following lemma is valid by virtue of Remark 9.1 of [1].

LEMMA 1. Each Hermitian matrix-valued kernel A(t, s) is expanded in the sense of convergence in the norm of the space $L_2([a, b] \times [a, b]; M)$ in the series

$$A(t, s) = \sum_{\nu=1}^{m} \lambda_\nu \begin{pmatrix} \varphi_1^{(\nu)}(t) \\ \vdots \\ \varphi_n^{(\nu)}(t) \end{pmatrix} (\bar{\varphi}_1^{(\nu)}(s) \ldots \bar{\varphi}_n^{(\nu)}(s)), \quad m \leqslant \infty, \tag{1}$$

where $\{\varphi^{(\nu)}(t) = (\varphi_i^{(\nu)}(t))\}$ is an orthonormal sequence of vector-valued eigenfunctions from $L_2([a, b]; E_n)$, $\{\lambda_\nu\}$ is the sequence of the corresponding nonzero eigenvalues of the Fredholm integral operator A, generated by the matrix-valued kernel A(t, s); in addition, the elements of $\{\lambda_\nu\}$ are disposed in the order of decreasing modulus.

LEMMA 2. If operator A, generated by a continuous Hermitian matrix-valued kernel A(t, s), is nonnegative, then series (1) is uniformly convergent, i.e.,

$$\lim_{N\to m} \sup_{(t, s)\in[a, b]\times[a, b]} \| A(t, s) - A^{(N)}(t, s) \|_M = 0,$$

where $A^{(N)}(t, s)$ denotes the N-th partial sum of the series (1).

Lemma 2 is a generalization of the Mercer theorem [2] and is proved by analogous method.

Let us consider the following system of Volterra integral equations of first kind:

$$Ku \equiv \int_a^t K(t, s)\, u(s)\, ds = f(t), \quad t \in [a, b], \tag{2}$$

where $K(t, s) = (K_{ij}(t, s)) \in L_2(G; M)$, $G = \{a < s < t < b\}$, and $u(t) = (u_i(t))$, $f(t) = (f_i(t)) \in L_2([a, b]; E_n)$. Let the following conditions be fulfilled:

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